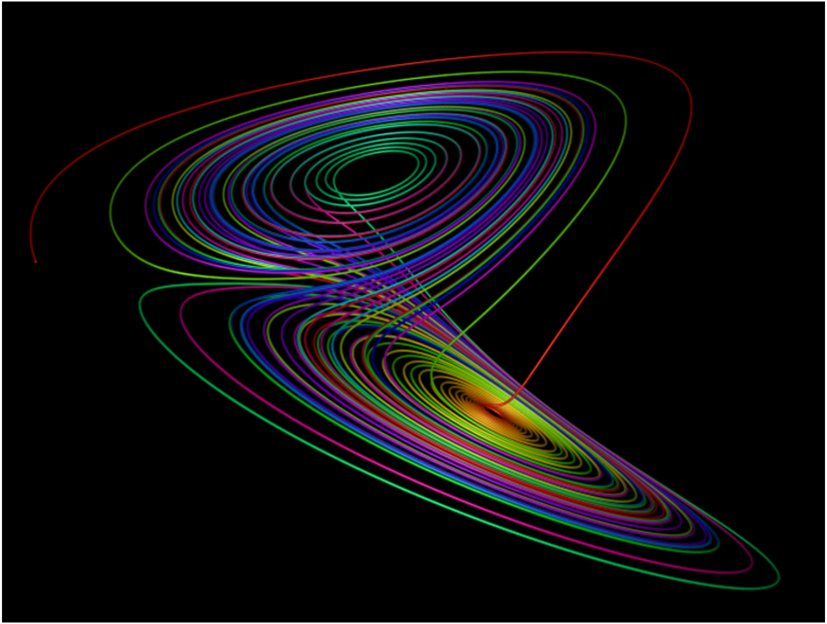
Physics 584 Computational Methods

The Lorenz Equations and Numerical Simulations of Chaos

Ryan Ogliore

April 25th, 2016




STRANGE,
YET ATTRACTIVE

Lecture Outline

Homework Review

Chaotic Dynamics

Lyapunov Exponent

Lyapunov Exponent of Lorenz Equations

Chaotic motion in the Solar System

Lecture Outline

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Chaotic Dynamics

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Homework (continued)

Edward Lorenz, a meterologist, created a simplified mathematical model for nonlinear atmospheric thermal convection in 1962. Lorenz's model frequently arises in other types of systems, e.g. dynamos and electrical circuits. Now known as the Lorenz equations, this model is a system of three ordinary differential equations:

$$\begin{aligned} \frac{dx}{dt} &= \sigma(y - x), \\ \frac{dy}{dt} &= x(\rho - z) - y, \\ \frac{dz}{dt} &= xy - \beta z. \end{aligned}$$

Note the last two equations involve quadratic nonlinearities. The intensity of the fluid motion is parameterized by the variable x; y and z are related to temperature variations in the horizontal and vertical directions.

Homework (continued)

Use Matlab's RK4 solver `ode45` to solve this system of ODEs with the following starting points and parameters.

1. With $\sigma = 1$, $\beta = 1$, and $\rho = 1$, solve the system of Lorenz Equations for $x(t = 0) = 1$, $y(t = 0) = 1$, and $z(t = 0) = 1$. Plot the orbit of the solution as a three-dimensional plot for times 0–100.
2. For the Earth's atmosphere reasonable values are $\sigma = 10$ and $\beta = 8/3$. Also set $\rho = 28$; and using starting values: $x(t = 0) = 5$, $y(t = 0) = 5$, and $z(t = 0) = 5$; solve the system of Lorenz Equations for $t = [0, 20]$. Plot the orbit of the solution as a three-dimensional plot for t=0–20. Also plot z vs. x. Do any of the orbits that appear to overlap in this plot actually overlap when viewed in the three-dimensional plot?
3. Plot x, y, and z vs. time on one graph using Matlab's `subplot` function.

Homework (continued)

4. Use the same parameters as in #2 but add a very small number (e.g. 10^{-6}) to one of the starting values. Plot x, y, and z vs. time for *both* of these curves (one red, one blue). Solve the equation for longer times to see when the two solutions diverge from each other.
5. Find a value of ρ (while keeping $\sigma = 10$ and $\beta = 8/3$) such that the solution does not depend sensitively on the initial values. Plot both curves for x, y, and z vs. time as you did in #4.
6. For ρ=70, $\sigma = 10$, $\beta = 8/3$, initial starting value (5,5,5), over a time range 0–50, calculate and plot one solution using the default maximum step size for `ode45`: $0.1 \times (t_{\text{final}} - t_{\text{initial}})$, and another solution for $1/1000^{\text{th}}$ of the default. Is this behavior related to the sensitivity on initial starting values you explored in #4?

Homework

The homeworks I have looked at so far were very good.

Question #2: Do any of the orbits that appear to overlap in this plot actually overlap when viewed in the three-dimensional plot?

The *Existence and Uniqueness Theorem* for systems of differential equations guarantees a unique solution for each set of initial conditions

We will look into some of the other aspects of the Lorenz equations in today's lecture.

Lecture Outline

Lorenz Equations

Lorenz Equations:

$$\begin{aligned}\frac{dx}{dt} &= \sigma(y - x),\\ \frac{dy}{dt} &= x(\rho - z) - y,\\ \frac{dz}{dt} &= xy - \beta z.\end{aligned}$$

You explored the behavior of the solution for these equations with various parameters and starting conditions.

Some History

- The concept of the “Clockwork Universe” was accepted after Isaac Newton’s laws of motion, and the improved analytical techniques for finding the equations for a system by Lagrange and Hamilton.
- Such a universe is completely determined by its initial conditions to evolve predictably with time (particularly championed by Laplace).
- Poincaré was the first person to see that Newton’s laws of motion, in fact, predicted chaos all along.

Henri Poincaré

Some History

- In 1886, in honor of his 60^{th} birthday, Oscar II, King of Sweden, established a prize for anyone who could find a solution to one of four oustanding questions in mathematical physics
- The problems were announced in *Acta Mathematica*, vol. 7, of 1885-1886. One problem was: "Given a system of arbitrarily many mass points that attract each other according to Newton's law, under the assumption that no two points ever collide, try to find a representation of the coordinates of each point as a series in a variable that is some known function of time and for all of whose values the series converges uniformly" (aka the n-body problem):

$$m_i \ddot{\mathbf{q}}_i = \sum_{j \neq i}^{n} \frac{G m_i m_j (\mathbf{q}_i - \mathbf{q}_j)}{|\mathbf{q}_i - \mathbf{q}_j|^3}, \; i = 1, \ldots, n$$

Some History

Poincaré proved that an analytical solution to the three-body problem was not possible.

"…it may happen that small differences in the initial conditions produce very great differences in the final phenomena. A small error in the former will produce an enormous error in the latter. Prediction becomes impossible…" – Henri Poincaré, 1892

Some History

- Poincaré discovered **deterministic chaos**
- However, it was overshadowed by quantum mechanics and relativity
- …until 1960, when computer simulations of simple systems of differential equations (e.g. the Lorenz equations) showed that even very simple systems can become chaotic

What is Deterministic Chaos?

- Each time step depends only on the motion at previous times in a well-defined way
- The “chaos” comes from the fact that long-term prediction is impossible without perfect knowledge of the initial conditions
- This has nothing to do with random noise, though the motion can look like random noise
- For chaotic motion, two trajectories that are initially arbitrarily close in phase space will diverge exponentially in time from each other
- All memory that the two trajectories started out close is lost
- **Exponential divergence** is key. For nonchaotic motion, nearby trajectories diverge at most linearly with time

Can we understand **exponential divergence**: how two trajectories diverge from a starting point?

This will tell us *if the system is chaotic* and *how long we are able to accurately predict the evolution of the system*.

This has many applications in real-world problems. For example, if the solar system is chaotic, how long can we predict that the Earth is safe from being hit by a rogue planet/asteroid, or from being ejected from the solar system?

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Lyapunov Exponent

Lyapunov exponents are characteristic quantities of dynamical systems. They parameterize this *exponential divergence*. For a continuous-time dynamical system, the maximal Lyapunov exponent is defined as follows:

Consider a trajectory $x(t)$, $t \geq 0$ in phase space and a nearby trajectory $x(t)$+$\delta(t)$, where $\delta(t)$ is a vector with infinitesimal initial length. As the system evolves, track how $\delta(t)$ changes. The *maximal* Lyapunov exponent of the system is the number λ, if it exists, such that:

$$|\delta(t)| \approx |\delta(0)|e^{\lambda t}$$

Lyapunov Exponent

There are as many Lyapunov exponents as there are dimensions in the phase space (e.g. for the Lorenz equations, with motion in x, y, and z, there are three Lyapunov exponents).

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The maximal Lyapunov exponent (MLE) is the most important one because it determines the predictability of the dynamical system. **A positive MLE is usually taken as an indication that the system is chaotic** (provided some other conditions are met, e.g., phase space compactness). The MLE will determine the separation between trajectories (the effects of the other exponents will be quickly overwhelmed).

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Local Lyapunov exponents estimate the local predictability around a point x_0 in phase space. These are easier to calculate than the global Lyapunov exponents. The local Lyapunov exponents are the eigenvalues of the Jacobian at x_0.

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Sensitivity on Initial Conditions for the Lorenz Equations

Lorenz Equations:

$$\begin{aligned} x' &= \sigma(y - x), \\ y' &= x(\rho - z) - y, \\ z' &= xy - \beta z. \end{aligned}$$

As you saw in the homework, an arbitrary starting point can diverge greatly from a very close neighboring starting point.

Sensitivity on Initial Conditions for the Lorenz Equations

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But you also found values of ρ where this was not the case. So the Lorenz equations can lend themselves to chaos or non-chaos depending on the parameters.

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Is there any way to predict this, analytically from the equations of the system?

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Let's explore this by first looking at *critical points* (also called fixed points).

Critical Points

Lorenz Equations:

$$\begin{aligned} x' &= \sigma(y - x), \\ y' &= x(\rho - z) - y, \\ z' &= xy - \beta z. \end{aligned}$$

Critical points are points in phase space that do not evolve. We will investigate the *stability* of fixed points because this will help us understand the general solutions of the system.

First, we find the **critical points** where $x' = y' = z' = 0$.

Critical Points

Matlab code to find critical points (symbolic solution):

```
clear all
close all

syms x y z xp yp zp
syms sigma_ positive
syms rho_ positive
syms beta_ positive

[x_c,y_c,z_c,~,conditions]=solve([0==sigma_*(y-x);0==x*(rho_-z)-y;0==x*y-beta_*z],...
    'ReturnConditions', true, 'Real', true);

disp([x_c,y_c,z_c,conditions]);

% >>lyapunov_example
% [                        0,                         0,      0, rho < 1 | 1 <= rho]
% [  beta^(1/2)*(rho - 1)^(1/2),  beta^(1/2)*(rho - 1)^(1/2), rho - 1,          1 <= rho]
% [ -beta^(1/2)*(rho - 1)^(1/2), -beta^(1/2)*(rho - 1)^(1/2), rho - 1,          1 <= rho]
```

Critical Points

Lorenz Equations:

$$x' = \sigma(y - x),$$
$$y' = x(\rho - z) - y,$$
$$z' = xy - \beta z.$$

Critical points ($x' = y' = z' = 0$):

1. $(0, 0, 0)$
2. $(\sqrt{\beta(\rho - 1)}, \sqrt{\beta(\rho - 1)}, \rho - 1)$
3. $(-\sqrt{\beta(\rho - 1)}, -\sqrt{\beta(\rho - 1)}, \rho - 1)$

if $\rho > 1$ and only $(0, 0, 0)$ for $\rho < 1$.

Critical Points

Are these critical points are stable (non-chaotic)? Meaning, if we start close, but not exactly on, the critical point, the solution will evolve toward the fixed point.

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Attractor: A region in space that is invariant under the evolution of time and attracts most, if not all, nearby trajectories.
Strange attractor: An attractor that displays chaotic behavior (that is, high sensitivity to initial conditions).

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The Lyapunov exponent is the eigenvalue of the Jacobian of the system.

Calculating Lyapunov Exponent

The Jacobian matrix of the Lorenz system at the critical point $(0, 0, 0)$ is:

$$\mathbf{J} = \frac{d\mathbf{f}}{d\mathbf{x}} = \begin{bmatrix} \frac{\partial \mathbf{f}}{\partial x_1} & \cdots & \frac{\partial \mathbf{f}}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

$$\mathbf{J_1} = \begin{bmatrix} -\sigma & \sigma & 0 \\ \rho & -1 & 0 \\ 0 & 0 & -\beta \end{bmatrix}$$

Which has the characteristic polynomial:

$$\lambda^3 + (\sigma + 1 + \beta)\lambda^2 + (\beta(\sigma + 1) + \sigma(1 - \rho))\lambda + \beta\sigma(1 - \rho) \quad (1)$$

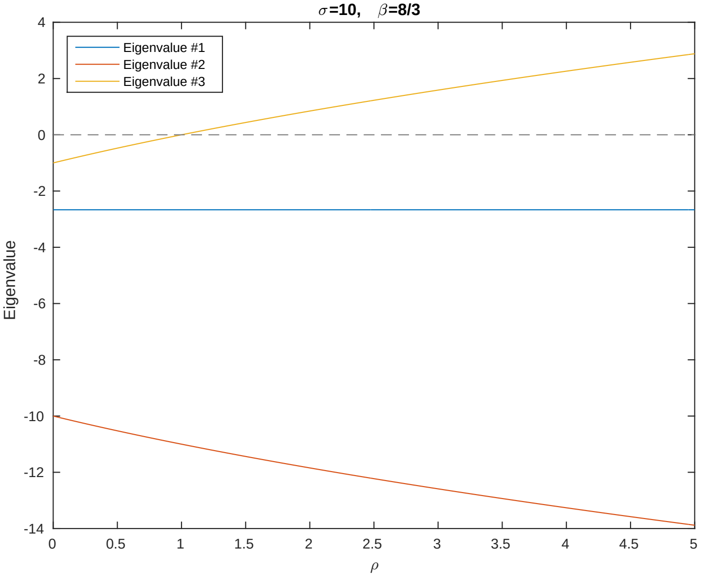
Eigenvalues are the roots of this polynomial:
$-\beta, -\sigma/2 \pm \sqrt{4\rho\sigma - 2\sigma + \sigma^2 + 1}/2 - 1/2$

Calculating Lyapunov Exponent

Matlab code to calculate eigenvalues of the Jacobian at the critical point $(0, 0, 0)$:

```
% Clear variables, close plots
clear all
close all

% Declare symbolic variables
syms x y z xp yp zp lambda
syms sigma_ positive
syms rho_ positive
syms beta_ positive

% Calculate the Jacobian
J=jacobian([sigma_*(y-x);x*(rho_-z)-y;x*y-beta_*z],[x,y,z]);

% Substitute the first critical point (0,0,0) into the Jacobian
J_1=subs(J,[x,y,z],[0,0,0])

% [ -sigma_, sigma_,       0]
% [    rho_,     -1,       0]
% [       0,      0, -beta_]

% Calculate the eigenvalues of this Jacobian
eigvals=eig(J_1);

%                                                         -beta_
%  - sigma_/2 - (4*rho_*sigma_ - 2*sigma_ + sigma_^2 + 1)^(1/2)/2 - 1/2
%    (4*rho_*sigma_ - 2*sigma_ + sigma_^2 + 1)^(1/2)/2 - sigma_/2 - 1/2

% Plot the three eigenvalues for sigma=10, beta=8/3, and a range of rho
rho_vals=0:50;
eigvals_=zeros(numel(rho_vals),numel(eigvals));
for ii=1:numel(rho_vals)
    eigvals_(ii,:)=double(subs(eigvals,[sigma_,beta_,rho_],[10,8./3.,rho_vals(ii)]));
end

plot(rho_vals,eigvals_)
xlabel('\rho')
ylabel('Eigenvalue')
legend('Eigenvalue #1','Eigenvalue #2','Eigenvalue #3','Location','NorthWest')
title('\sigma=10, \beta=8/3')
pdfname='eigenvals_lorenz_jacobian.pdf';
print('-dpdf',pdfname);
[~,~]=system(['pdfcrop ' pdfname ' ' pdfname]);
```

Calculating Lyapunov Exponent for (0,0,0)



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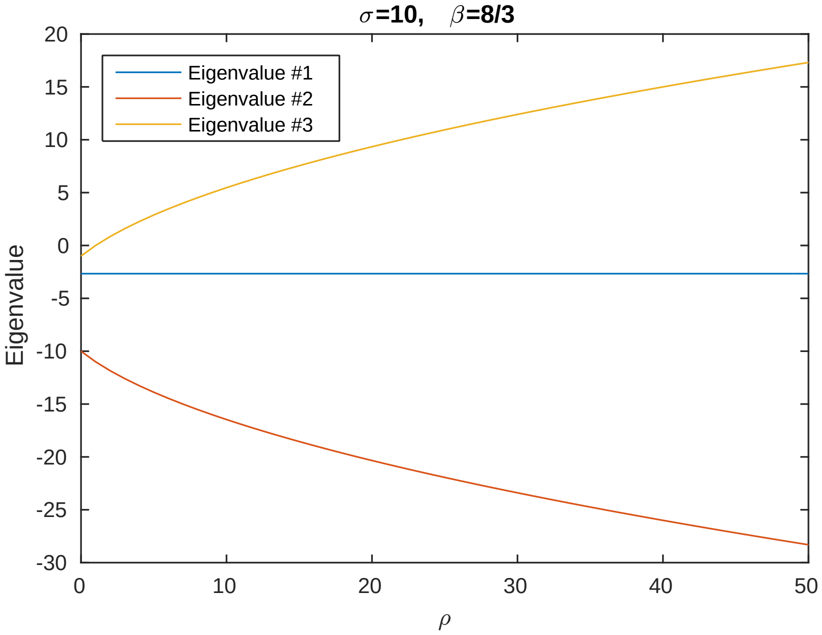
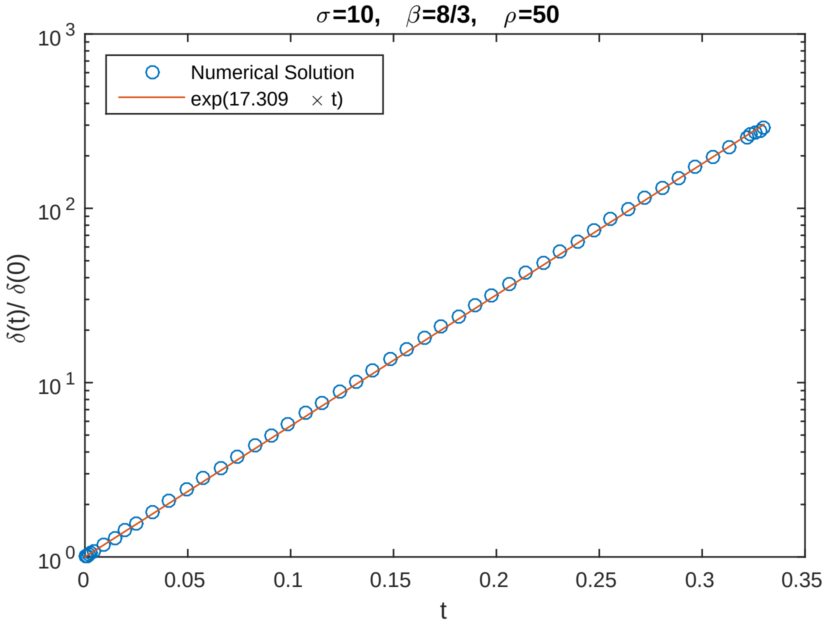


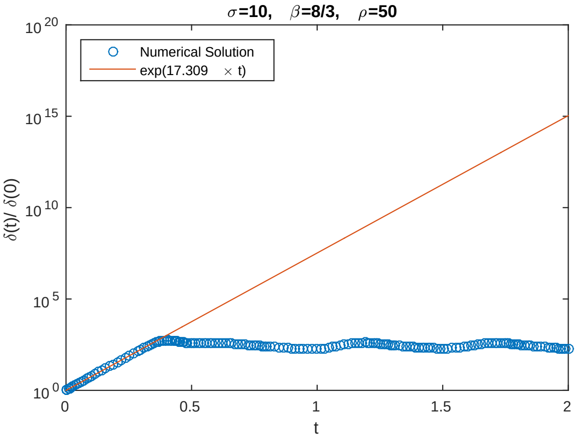
Lyapunov Exponent

Does the Lyapunov Exponent predict the time-dependent deviation between an orbit starting at the initial value $(0, 0, 0)$ and another orbit starting value very close to this? Let's look at ρ=50.

Lyapunov Exponent of Lorenz Equations

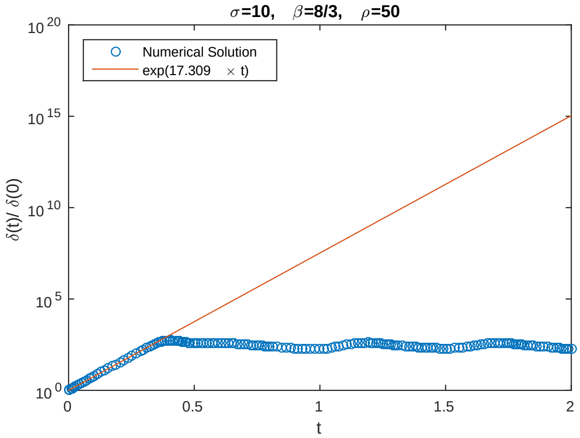


Lyapunov Exponent of Lorenz Equations



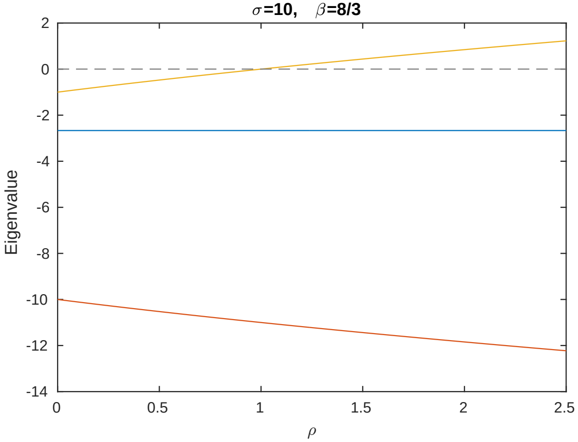
The solution to the Lorenz equations occupies a finite volume of phase space, so the two trajectories cannot deviate arbitrarily far apart. **The Lyapunov exponent does not tell the whole story.**

Lyapunov Exponent of Lorenz Equations



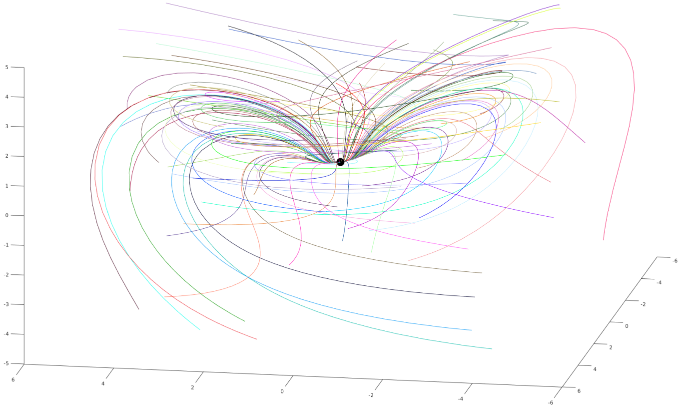
Analogously, a double pendulum can appear to behave chaotically but its motion is still constrained by its Hamiltonian. It can only occupy a finite region of phase space.

Lyapunov Exponent of Lorenz Equations



For $\rho < 1$ at critical point $(0, 0, 0)$, all three Lyapunov exponents are negative: **no chaos**. This means the origin is a stable critical point for $\rho < 1$. That is, the origin is a "sink" and all solutions are drawn to it.

Lyapunov Exponent of Lorenz Equations



$$\sigma = 10,\ \beta = 8/3,\ \rho = 0.5$$

Lyapunov Exponent of Lorenz Equations

What about the other two critical points:

$(\pm\sqrt{\beta(\rho-1)}, \pm\sqrt{\beta(\rho-1)}, \rho-1) = C^{\pm}$?

Are these stable (non-chaotic) or unstable (chaotic)? We'll need to look at their Lyapunov exponents.

Lyapunov Exponent of Lorenz Equations

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When we do this, we find that for some values of ρ, the eigenvalues have an imaginary component!

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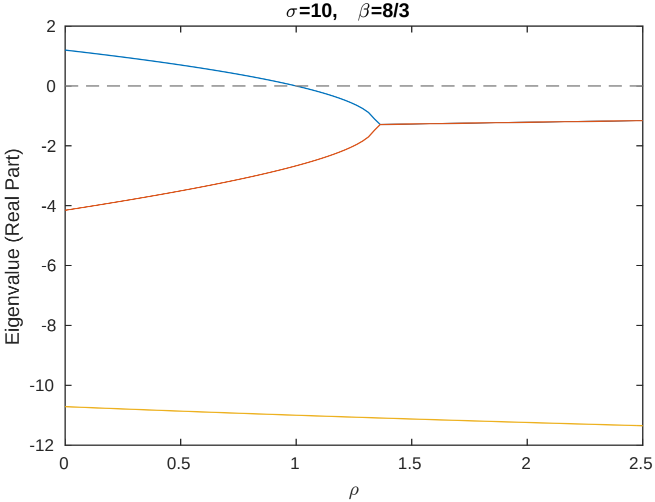
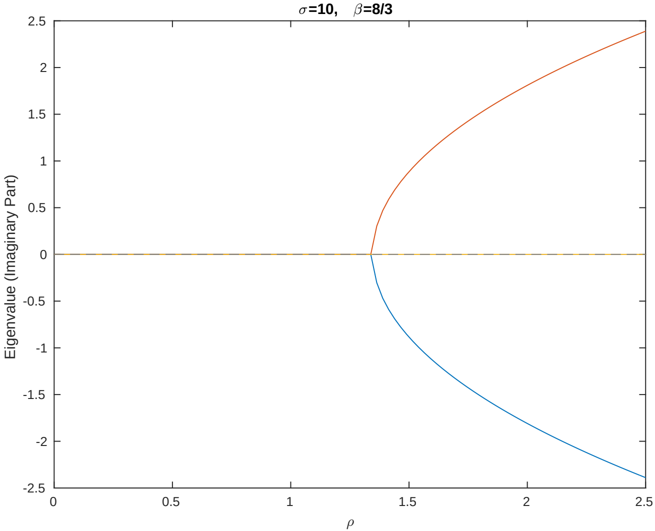
When we do this, we find that for some values of ρ, the eigenvalues have an imaginary component! There are also bifurcations and other interesting behavior.

Lyapunov Exponent of Lorenz Equations: $C^{\pm}$



Lyapunov Exponent of Lorenz Equations: $C^{\pm}$



Lyapunov Exponent of Lorenz Equations

What does the imaginary eigenvalue mean for the orbits?

Lyapunov Exponent of Lorenz Equations

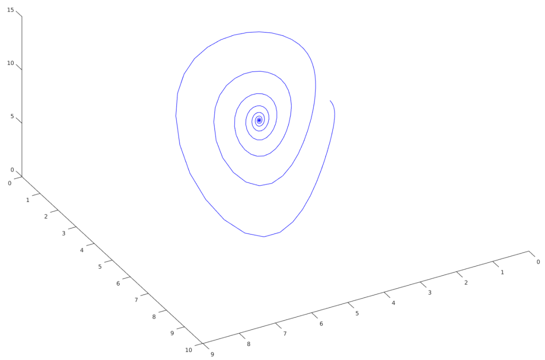
What does the imaginary eigenvalue mean for the orbits?

Recall (the deviation between two orbits):

$$|\delta(t)| \approx |\delta(0)|e^{\lambda t}$$

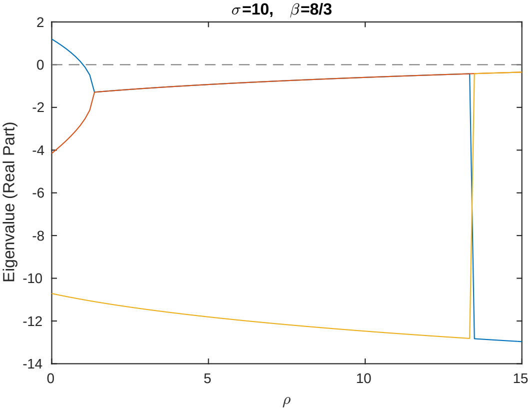
So negative eigenvalues would mean *oscillatory* orbits (they begin at ρ=1.346) with a frequency given by the imaginary component.

Lyapunov Exponent of Lorenz Equations



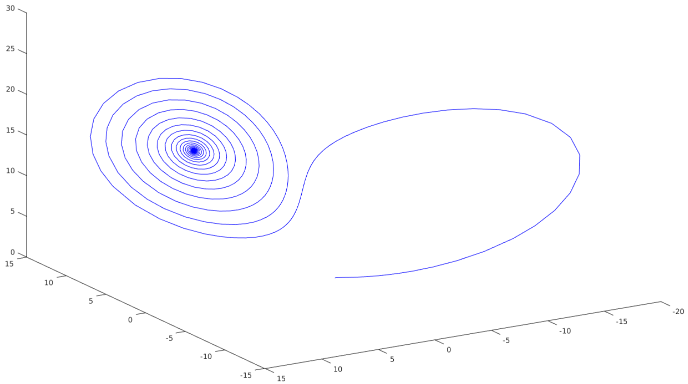
$\sigma = 10$, $\beta = 8/3$, $\rho = 10$. Orbits spiral towards either C^+ or C^-.

Lyapunov Exponent of Lorenz Equations

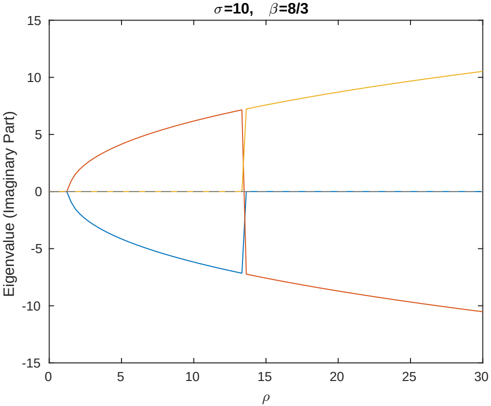


Plotting the eigenvalues for higher ρ, we see that at ρ=13.926 the value of the largest eigenvalue switches to a different eigenvector. This eigenvector has a different direction, and instead, a point closer to C^+ will eventually settle at C^- (and vice-versa).

Lyapunov Exponent of Lorenz Equations

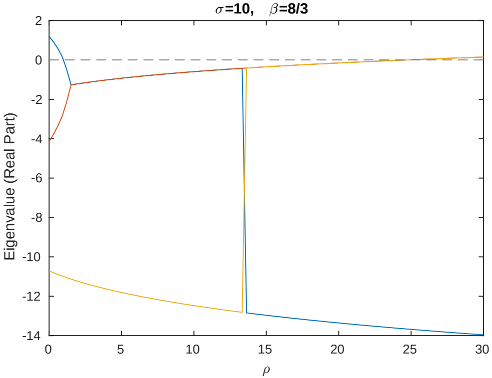


Lyapunov Exponent of Lorenz Equations



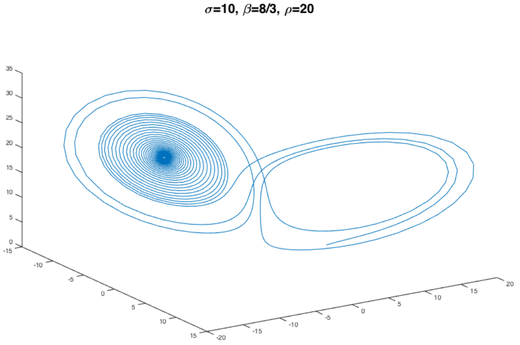
From ρ=13–24, the imaginary part of the eigenvalues increase, meaning the frequency of orbits around $C^{\pm}$ increase. Solutions oscillate between C^{+} and C^{-} many times before finally spiraling into them.

Lyapunov Exponent of Lorenz Equations



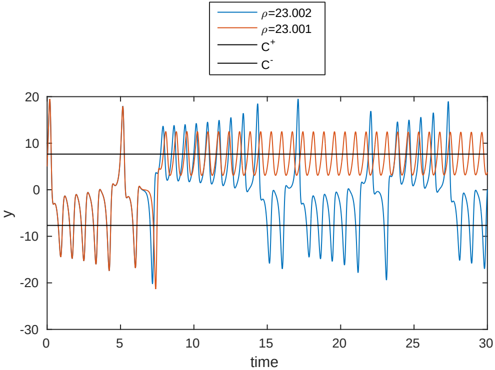
The real part of the eigienvalues for $C^{\pm}$ are still negative, so the solution will settle into either $C^{\pm}$ eventually, but the time it takes to do so can vary sensitively on the initial conditions.

Problem #5 Solution Some of You Found



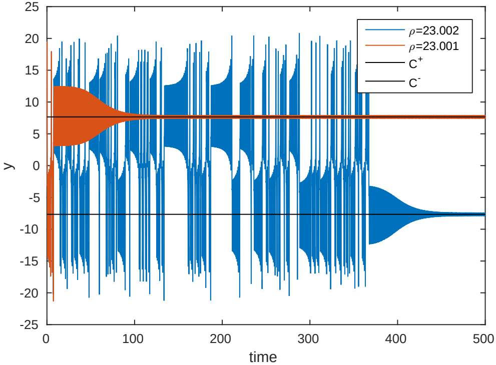
Problem #5: “Find a value of ρ (while keeping σ = 10 and β = 8/3) such that the solution does not depend sensitively on the initial values.”. Initial value = (5,5,5). Solution spirals into C^{-}=(-7.1,-7.3,19.0).

Lyapunov Exponent of Lorenz Equations



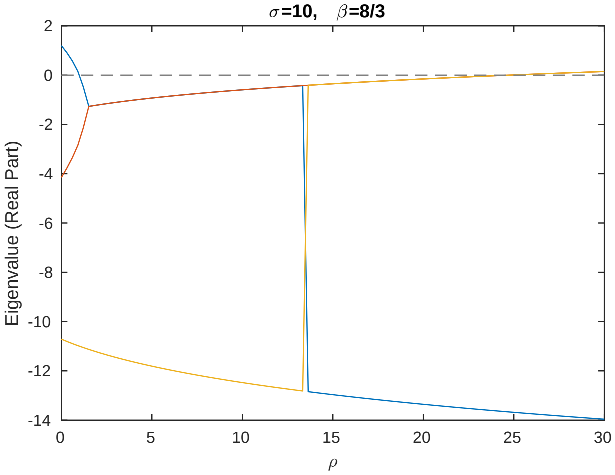
Some of you found interesting behavior in this region. Setting ρ=23.001 and ρ=23.002 and integrating $t = 0 - -30$ appeared to straddle a transition into chaos.

Lyapunov Exponent of Lorenz Equations



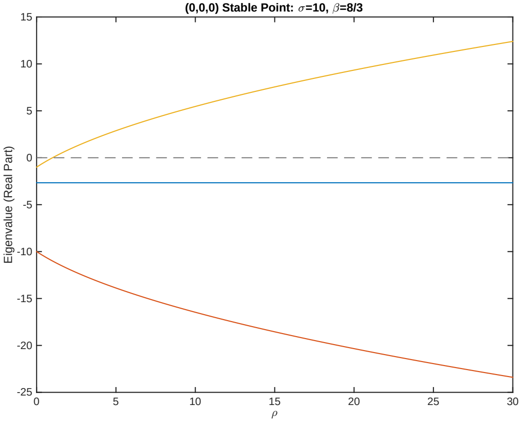
But if we integrate out to longer times, we see that each of these ρ values eventually spirals into $C^{\pm}$ (as they should because the eigenvalues are negative), but the time a solution takes to do so (and whether it goes to C^{+} or C^{-} can vary hugely with small changes in ρ, initial conditions, or integration time step! This is a fascinating region!

Lyapunov Exponent of Lorenz Equations



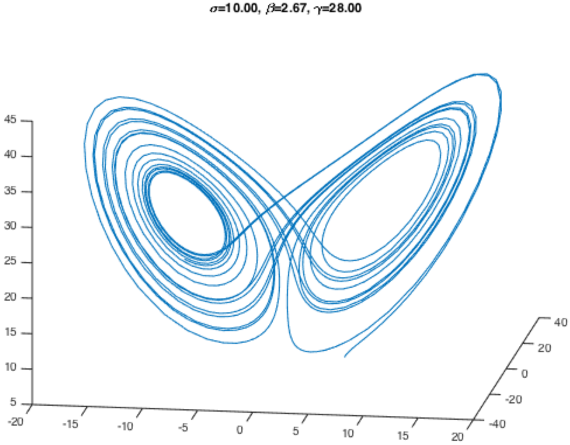
Around 24.74, one eigenvalue of $C^{\pm}$ turns positive again, so these attractors become chaotic, and because they have nonzero imaginary components, the orbits are spirals.

Lyapunov Exponent of Lorenz Equations



The origin (0,0,0) remains the same: unstable with interesting manifolds that depend on ρ (no-spiral because it does not have an imaginary component in its eigenvalues).

Lyapunov Exponent of Lorenz Equations



In this regime $C^{\pm}$ are strange attractors (displaying chaotic behavior). (This is the regime of the classic “Lorenz Attractor”).

Limit Cycle

A limit cycle is an isolated closed trajectory:

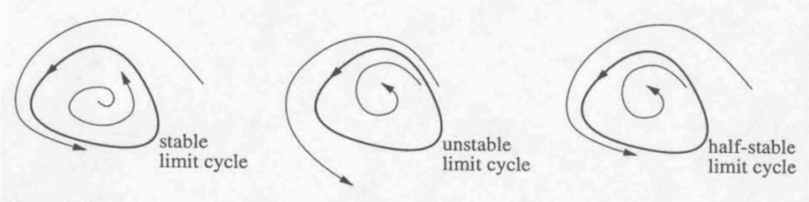


Figure 7.0.1

Lyapunov Exponent of Lorenz Equations

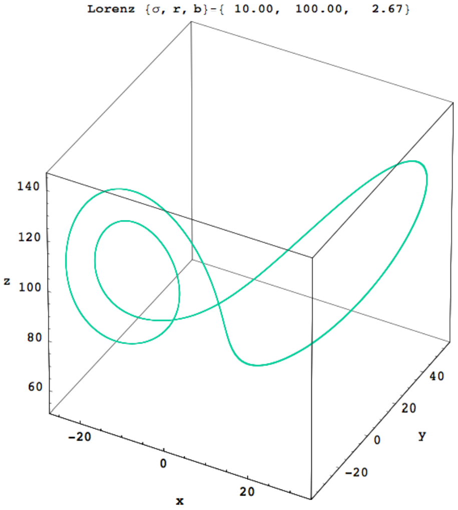
There are many surprises in the parameter space, and many cannot be easily predicted:

- Point and chaotic attractors for $24.06 < \rho < 24.74$
- Strange limit cycle behavior at Hopf bifurcation
- The return of a global attracting limit cycle for intervals at large ρ
- Lots more…

Exploring the phase space of the Lorenz equations is like a walk in the jungle.

Limit Cycle

Attractor is a stable limit cycle for ρ=100



Lecture Outline

Homework Review

Chaotic Dynamics

Lyapunov Exponent

Lyapunov Exponent of Lorenz Equations

Chaotic motion in the Solar System

Chaotic motion in the Solar System

One of the reasons the King Oscar wanted to know the solution to the n-body problem was to determine if the solar system is stable.

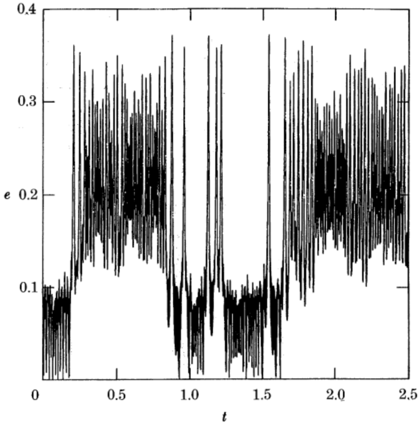


FIGURE 5. Eccentricity of a typical chaotic trajectory over a longer time interval. The time is now measured in millions of years. Bursts of high-eccentricity behaviour are interspersed with intervals of irregular low-eccentricity behaviour, broken by occasional spikes.

Chaotic motion in the Solar System

- Asteroids in orbital resonances with Jupiter can be sent into Earth-crossing orbits.
- Pluto has a Lyapunov exponent of 1/20 Myr^{-1}. This implies a *Lyapunov time scale* of $\sim$20 Myr before neighboring orbits diverge significantly (i.e. it's not possible to determine Pluto's orbit precisely $\sim$100 Myr from now).
- A rocket launch will change the Earth's position by $\sim 0.5^\circ$ after 100 Myr.
- But the Solar Sytem's dynamics are governed by a Hamiltonian, so chaos can only be local (a specific area of phase space).
- Arnold diffusion (nonconservation of invariant quantities) would allow for local chaos to diffuse throughout the phase space.
- Arnold diffusion would cause more catastrophic chaos, like ejection of planets, but the timescales for this appear to be large.