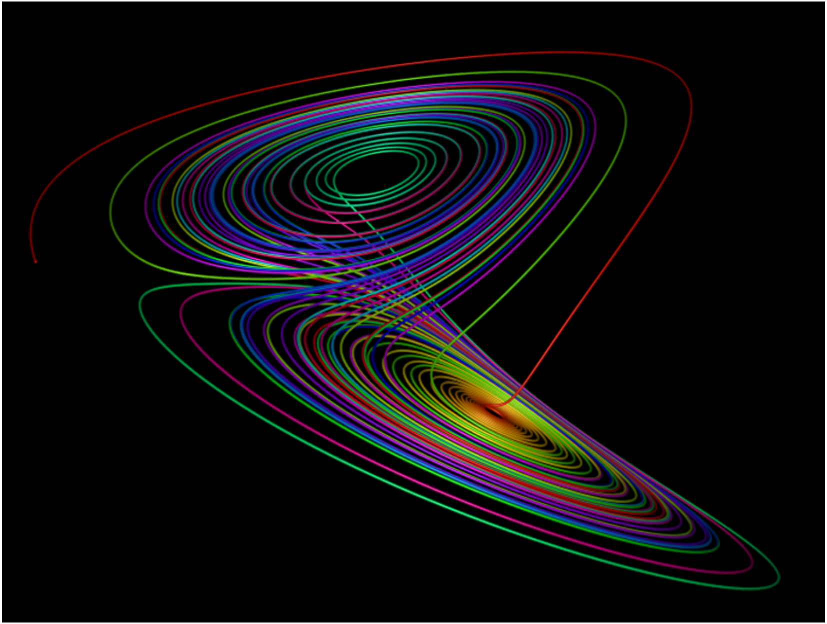
Physics 584 Computational Methods

The Lorenz Equations and Numerical Simulations of Chaos

Ryan Ogliore

April 18th 2019




STRANGE,
YET ATTRACTIVE

Lecture Outline

Lorenz Equations in an Analog Electronic Circuit

Homework Review

Chaotic Dynamics

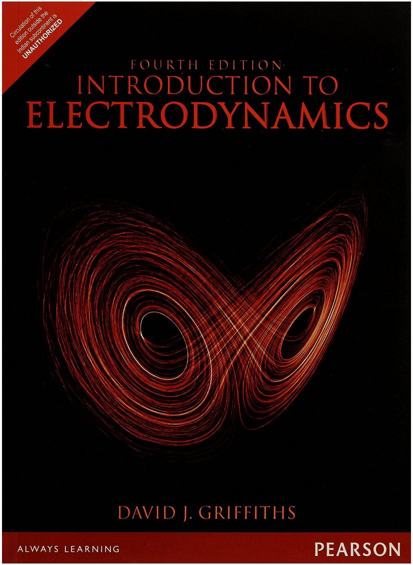
Lyapunov Exponent

Lyapunov Exponent of the Lorenz Equations

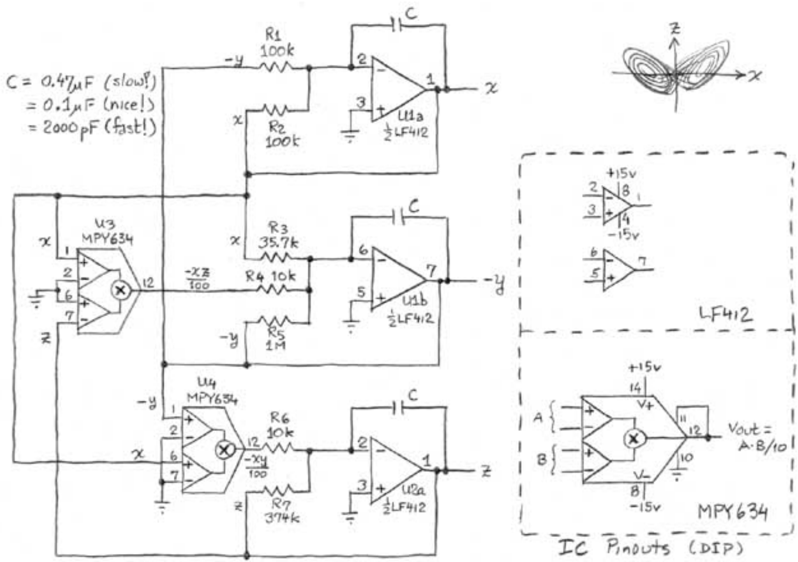
Chaotic motion in the Solar System

Lecture Outline

Lorenz Equations in an Analog Electronic Circuit



Lorenz Equations in an Analog Electronic Circuit



Lorenz Equations

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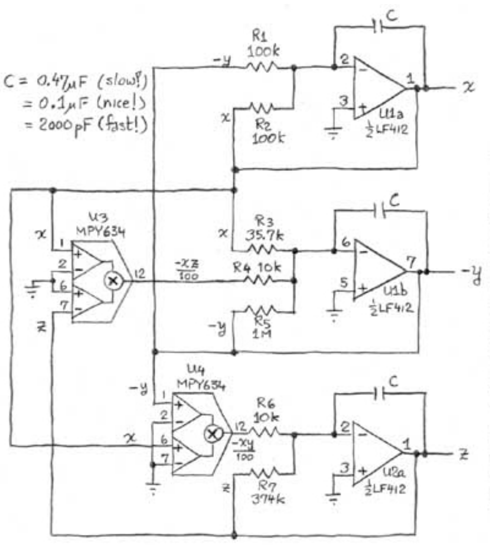

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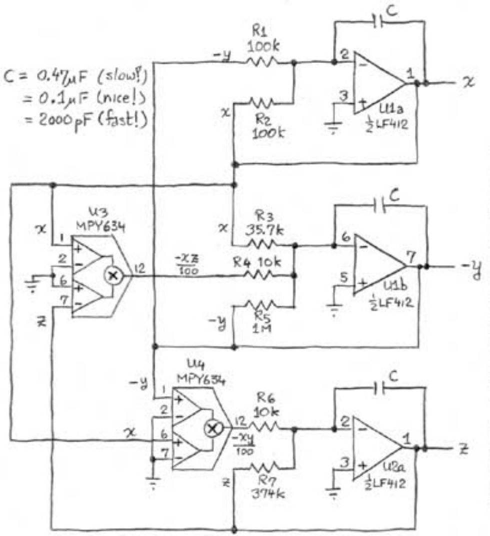

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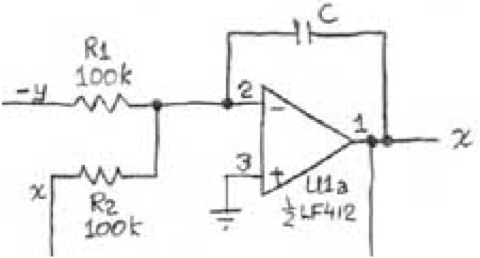

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- Terms that make up the derivatives are summed at op-amp input
- Resistor terms are scaled to 1 MΩ, so x is calculated:

$$x = -\int \frac{1\ \mathrm{M}\Omega}{100\ \mathrm{k}\Omega}(-y + x)$$

$$= -\int 10(x - y) = \int 10(y - x)$$

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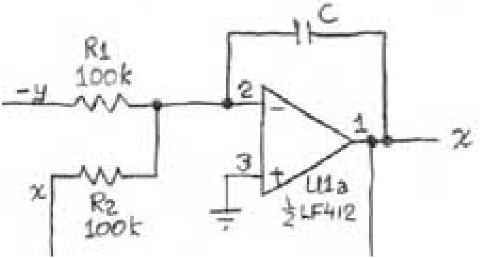

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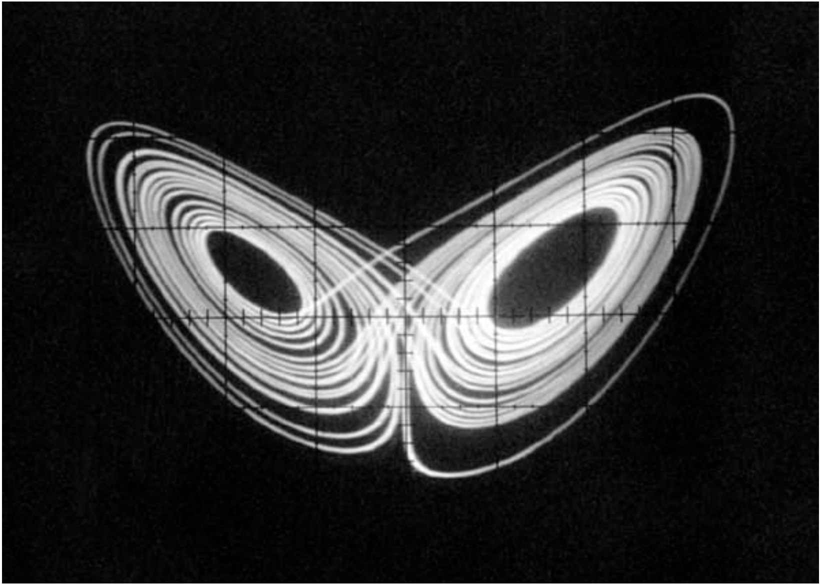
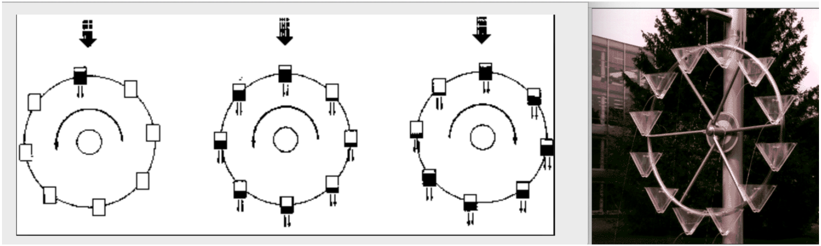
- …and likewise for dy/dt and dz/dt in the Lorenz equations

Lorenz Equations in an Analog Electronic Circuit



Lorenz Equations: Chaotic Waterwheel



The leaky waterwheel follows the Lorenz equations and display analogous behavior.

Lecture Outline

- Lorenz Equations in an Analog Electronic Circuit
- Homework Review
- Chaotic Dynamics
- Lyapunov Exponent
- Lyapunov Exponent of the Lorenz Equations
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Homework (continued)

Edward Lorenz, a meterologist, created a simplified mathematical model for nonlinear atmospheric thermal convection in 1962. Lorenz's model frequently arises in other types of systems, e.g. dynamos and electrical circuits. Now known as the Lorenz equations, this model is a system of three ordinary differential equations:

$$\begin{aligned}
\frac{dx}{dt} &= \sigma(y - x), \\
\frac{dy}{dt} &= x(\rho - z) - y, \\
\frac{dz}{dt} &= xy - \beta z.
\end{aligned}$$

Note the last two equations involve quadratic nonlinearities. The intensity of the fluid motion is parameterized by the variable x; y and z are related to temperature variations in the horizontal and vertical directions.

Homework (continued)

Use Matlab's RK4 solver `ode45` to solve this system of ODEs with the following starting points and parameters.

1. With $\sigma = 1$, $\beta = 1$, and $\rho = 1$, solve the system of Lorenz Equations for $x(t = 0) = 1$, $y(t = 0) = 1$, and $z(t = 0) = 1$. Plot the orbit of the solution as a three-dimensional plot for times 0–100.
2. For the Earth's atmosphere reasonable values are $\sigma = 10$ and $\beta = 8/3$. Also set $\rho = 28$; and using starting values: $x(t = 0) = 5$, $y(t = 0) = 5$, and $z(t = 0) = 5$; solve the system of Lorenz Equations for $t = [0, 20]$. Plot the orbit of the solution as a three-dimensional plot for t=0–20. Also plot z vs. x. Do any of the orbits that appear to overlap in this plot actually overlap when viewed in the three-dimensional plot?
3. Plot x, y, and z vs. time on one graph using Matlab's `subplot` function.

Homework (continued)

4. Use the same parameters as in #2 but add a very small number (e.g. 10^{-6}) to one of the starting values. Plot x, y, and z vs. time for *both* of these curves (one red, one blue). Solve the equation for longer times to see when the two solutions diverge from each other.
5. Find a value of ρ (while keeping $\sigma = 10$ and $\beta = 8/3$) such that the solution does not depend sensitively on the initial values. Plot both curves for x, y, and z vs. time as you did in #4.
6. For ρ=70, $\sigma = 10$, $\beta = 8/3$, initial starting value (5,5,5), over a time range 0–50, calculate and plot one solution using the default maximum step size for `ode45`: $0.1 \times (t_{\text{final}} - t_{\text{initial}})$, and another solution for $1/1000^{\text{th}}$ of the default. Is this behavior related to the sensitivity on initial starting values you explored in #4?

Homework

`ode45`

- Matlab's `ode45` employs the Dormand-Prince method, a type of Runge-Kutta method
- This method computes 4^{th} and 5^{th} order solutions. **The difference between these solutions is then taken to be the error of the (fourth-order) solution**
- If the error is smaller than the tolerance, the step is successful
- If the error is larger than the *tolerance* (see `odeset`: `AbsTol` and `RelTol`) , the step is unsuccessful and the step size is decreased by an amount determined by the error/tolerance ratio
- This is an adaptive step-size integration algorithm. The user usually inputs a two-element vector for the time span over which to calculate the solution, the function returns a solution over it.
- `ode45` is an extremely accurate solver!

Homework: vectors

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- Matlab is optimized to do math operations on vectors and matrices (hence its name)
- Vectorizing code usually makes it easier to read and simpler to understand (but not always)
- Better to do operations explicitly rather than implicitly (e.g. time interpolation for output of ode45)
- Your code is a reflection of yourself, take pride in it!

Homework

Question #2: ***Do any of the orbits that appear to overlap in this plot actually overlap when viewed in the three-dimensional plot?***

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The *Existence and Uniqueness Theorem* for systems of differential equations guarantees a unique solution for each set of initial conditions:

Consider the initial value problem

$$y'(t) = f(t, y(t)), \qquad y(t_0) = y_0$$

If f is uniformly **Lipschitz continuous** in y and continuous in t, then for some value ε, there exists a unique solution $y(t)$ to the initial value problem on the interval $[t_0 - \varepsilon, t_0 + \varepsilon]$

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We will look into some of the other aspects of the Lorenz equations in today's lecture.

Lecture Outline

Lorenz Equations

Lorenz Equations:

$$\frac{dx}{dt} = \sigma(y - x),$$
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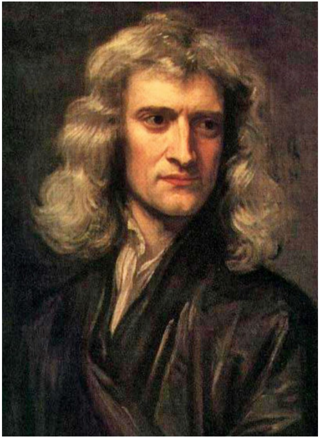
You explored the behavior of the solution for these equations with various parameters and starting conditions.

Some History

- The concept of the ***Clockwork Universe*** was accepted after Isaac Newton's laws of motion became canon (1690), and the improved analytical techniques for finding the equations for a system by Lagrange (1790) and Hamilton (1835).



Isaac Newton

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- Such a universe is completely determined by its initial conditions to evolve predictably with time (particularly championed by Laplace).

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- Such a universe is completely determined by its initial conditions to evolve predictably with time (particularly championed by Laplace).
- Poincaré was the first person to see that Newton’s laws of motion, in fact, predicted chaos all along.

Henri Poincaré

Some History

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- The problems were announced in *Acta Mathematica*, vol. 7, of 1885-1886. One problem was:

  Given a system of arbitrarily many point masses that attract each other according to Newton's laws, under the assumption that no two points ever collide, find a representation of the coordinates of each point as a series solution in a variable that is some known function of time and for all of whose values the series converges uniformly.

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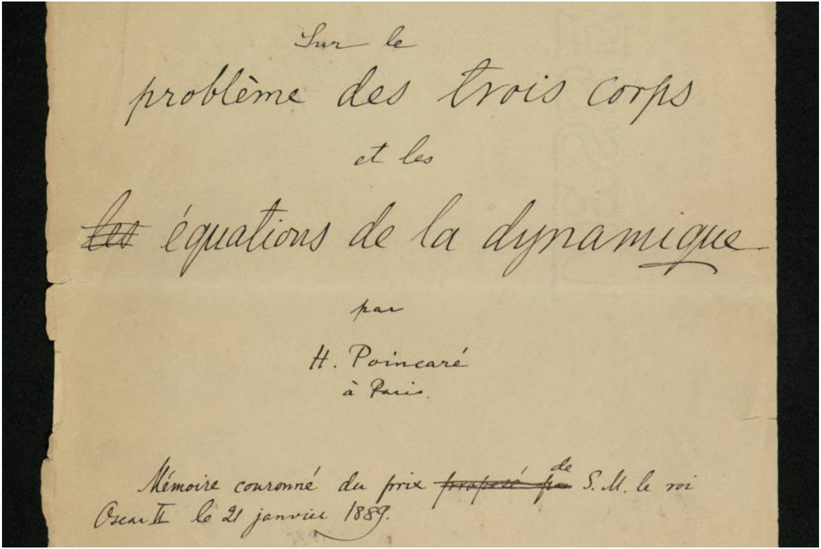
This is the n-body problem:

$$m_i \ddot{\mathbf{q}}_i = \sum_{j \neq i}^{n} \frac{G m_i m_j (\mathbf{q}_i - \mathbf{q}_j)}{|\mathbf{q}_i - \mathbf{q}_j|^3}, \; i = 1, \ldots, n$$

Some History

Poincaré proved that an analytical solution to the three-body problem was not possible.

"…it may happen that small differences in the initial conditions produce very great differences in the final phenomena. A small error in the former will produce an enormous error in the latter. Prediction becomes impossible…" – Henri Poincaré, 1892



Sur le

problème des trois corps

et les

~~les~~ équations de la dynamique

par

H. Poincaré

à Paris.

Mémoire couronné du prix ~~proposé par~~ de S. M. le roi Oscar II le 21 janvier 1889.

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- Poincaré discovered **deterministic chaos**
- However, it was overshadowed by quantum mechanics and relativity
- …until $\sim$1960, when computer simulations of simple systems of differential equations (e.g. the Lorenz equations) showed that even very simple systems can become chaotic
- Edward Lorenz: “*Chaos is when the present determines the future, but the approximate present does not approximately determine the future.*”

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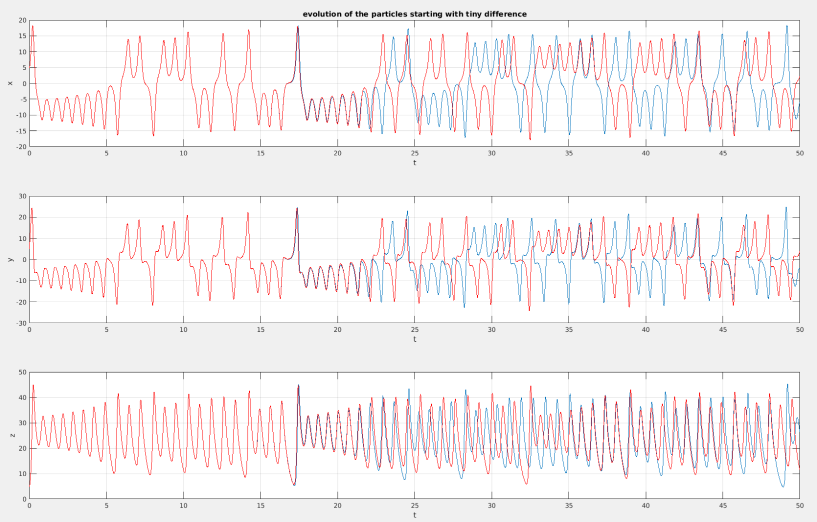
- **Exponential divergence** is key. For nonchaotic motion, nearby trajectories diverge at most *linearly* with time

What is Deterministic Chaos? (Problem #4 in HW)



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- Can we understand **exponential divergence**: how two trajectories diverge from a starting point?
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- This has many applications in real-world problems.
- For example, if the Solar System is chaotic, how long can we predict that the Earth is safe from being hit by a rogue planet/asteroid, or from being ejected from the Solar System?

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- As the system evolves, we track $\delta(t)$.
- The *maximal* Lyapunov exponent of the system is the number λ, if it exists, such that:

$$|\delta(t)| \approx |\delta(0)|e^{\lambda t}$$

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- The local Lyapunov exponents are the eigenvalues of the Jacobian of f (in $x' = f(x, t)$) at x_0.

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Sensitivity on Initial Conditions for the Lorenz Equations

- Lorenz Equations:
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As you saw in the homework, an arbitrary starting point can diverge greatly from a very close neighboring starting point.

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- But you also found values of ρ where this was not the case. So the Lorenz equations can lend themselves to chaos or non-chaos depending on the parameters.

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- Is there any way to predict this, *analytically*, from the equations of the system?
- Let's explore this by first looking at ***critical points*** (also called ***fixed points***).

Critical Points

$$x' = \sigma(y - x),$$
$$y' = x(\rho - z) - y,$$
$$z' = xy - \beta z.$$

Critical points are points in phase space that do not evolve. We will investigate the *stability* of fixed points because this will help us understand the general solutions of the system.

First, we find the **critical points** by setting $x' = y' = z' = 0$ (time derivatives are zero $\rightarrow$ no change in x, y, z with time).

Critical Points

Matlab code to find critical points (symbolic solution):

```
clearvars
close all

syms x y z xp yp zp
syms sigma_ positive
syms rho_ positive
syms beta_ positive

[x_c,y_c,z_c,~,conditions]=solve([0==sigma_*(y-x);0==x*(rho_-z)-y;0==x*y-beta_*z],...
    'ReturnConditions', true, 'Real', true);

disp([x_c,y_c,z_c,conditions]);

% >>lyapunov_example
% [                          0,                          0,       0, rho < 1 | 1 <= rho]
% [   beta^(1/2)*(rho - 1)^(1/2),  beta^(1/2)*(rho - 1)^(1/2), rho - 1,          1 <= rho]
% [ -beta^(1/2)*(rho - 1)^(1/2), -beta^(1/2)*(rho - 1)^(1/2), rho - 1,          1 <= rho]
```

Critical Points

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Critical points $(x' = y' = z' = 0)$:

1. $(0, 0, 0)$
2. $(\sqrt{\beta(\rho - 1)}, \sqrt{\beta(\rho - 1)}, \rho - 1)$
3. $(-\sqrt{\beta(\rho - 1)}, -\sqrt{\beta(\rho - 1)}, \rho - 1)$

…if $\rho \geq 1$

…and only $(0, 0, 0)$ for $\rho < 1$.

Critical Points

- **Stable:** if we start close, but not exactly on, the critical point, the solution will evolve toward the fixed point.

Critical Points

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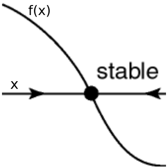
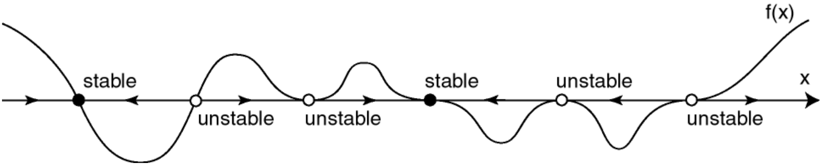
- *Are these critical points stable (i.e., non-chaotic)?*
- We can explore this by calculating the **Lyapunov exponents** of the system at these critical points.
- (The Lyapunov exponents are the eigenvalues of the Jacobian of the system.)

Stability of Critical Points



- Consider a 1-D system: $\frac{dx}{dt} = f(x)$
- Critical points: $f(x) = 0$
- Jacobian $J = f'(x)$ (slope of $f(x)$)
- Near stable point x_0, if $f(x) > 0$ for $x < x_0$: $\frac{dx}{dt}$ is also positive and you move to the right, back toward the critical point.
- …, if $f(x) < 0$ for $x > x_0$: $\frac{dx}{dt}$ is negative and you move to the left, also back toward the critical point.
- $\rightarrow$ the critical point is **stable** if $f'(x) < 0$

Frame Title



- Consider a 1-D system: $\frac{dx}{dt} = f(x)$
- Critical points: $f(x) = 0$
- Jacobian $J = f'(x)$ (slope of $f(x)$)
- Stable critical points $\rightarrow f'(x) < 0$
- Multidimensional case:
  Negative eigenvalues of Jacobian $\rightarrow$ time evolution points back to critical point (*stable critical point*)
  Positive eigenvalues $\rightarrow$ points away from critical point (*unstable critical point*)

Calculating Lyapunov Exponent

The Jacobian matrix of the Lorenz system at the critical point $(0, 0, 0)$:

$$\mathbf{J} = \frac{d\mathbf{f}}{d\mathbf{x}} = \begin{bmatrix} \dfrac{\partial \mathbf{f}}{\partial x_1} & \cdots & \dfrac{\partial \mathbf{f}}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial f_m}{\partial x_1} & \cdots & \dfrac{\partial f_m}{\partial x_n} \end{bmatrix}$$

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$$\mathbf{J_1} = \begin{bmatrix} -\sigma & \sigma & 0 \\ \rho & -1 & 0 \\ 0 & 0 & -\beta \end{bmatrix}$$

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Which has the characteristic polynomial (det $(\lambda I - \mathbf{J_1})$):

$$\lambda^3 + (\sigma + 1 + \beta)\lambda^2 + (\beta(\sigma + 1) + \sigma(1 - \rho))\lambda + \beta\sigma(1 - \rho)$$

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Eigenvalues (Lyapunov exponents) are the roots of this polynomial:

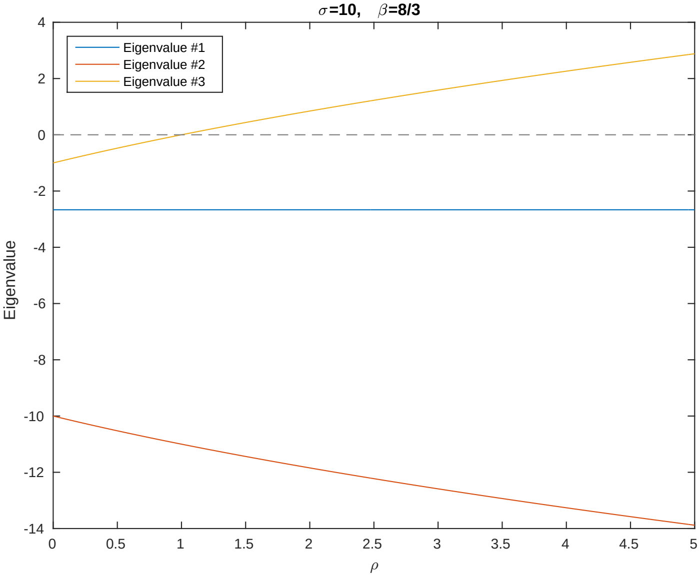
$-\beta, -\sigma/2 \pm \left(\sqrt{4\rho\sigma - 2\sigma + \sigma^2 + 1}\right)/2 - 1/2$

Calculating Lyapunov Exponent

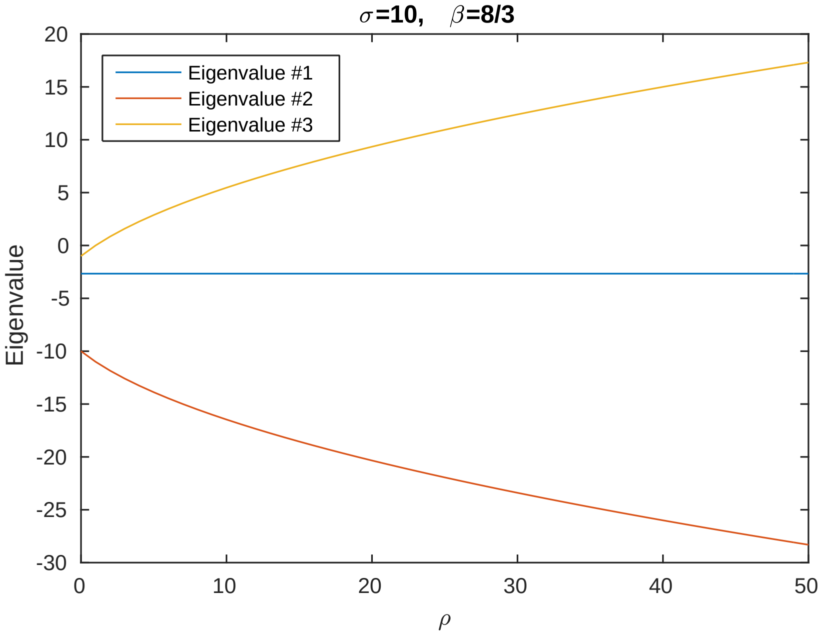
Matlab code to calculate eigenvalues of the Jacobian (Lyapunov exponents) at the critical point $(0, 0, 0)$:

```
% Clear variables, close plots
clearvars
close all

% Declare symbolic variables
syms x y z xp yp zp lambda
syms sigma_ positive
syms rho_ positive
syms beta_ positive

% Calculate the Jacobian
J=jacobian([sigma_*(y-x);x*(rho_-z)-y;x*y-beta_*z],[x,y,z]);

% Substitute the first critical point (0,0,0) into the Jacobian
J_1=subs(J,[x,y,z],[0,0,0])

% [ -sigma_, sigma_,      0]
% [    rho_,     -1,      0]
% [       0,      0, -beta_]

% Calculate the eigenvalues of this Jacobian
eigvals=eig(J_1);

%                                                   -beta_
% - sigma_/2 - (4*rho_*sigma_ - 2*sigma_ + sigma_^2 + 1)^(1/2)/2 - 1/2
%   (4*rho_*sigma_ - 2*sigma_ + sigma_^2 + 1)^(1/2)/2 - sigma_/2 - 1/2

% Plot the three eigenvalues for sigma=10, beta=8/3, and a range of rho
rho_vals=0:50;
eigvals_=zeros(numel(rho_vals),numel(eigvals));
for ii=1:numel(rho_vals)
   eigvals_(ii,:)=double(subs(eigvals,[sigma_,beta_,rho_],[10,8./3.,rho_vals(ii)]));
end

plot(rho_vals,eigvals_)
xlabel('\rho')
ylabel('Eigenvalue')
legend('Eigenvalue #1','Eigenvalue #2','Eigenvalue #3','Location','NorthWest')
title('\sigma=10, \beta=8/3')
pdfname='eigenvals_lorenz_jacobian.pdf';
print('-dpdf',pdfname);
[~,~]=system(['pdfcrop ' pdfname ' ' pdfname]);
```

Calculating Lyapunov Exponent for (0,0,0), ρ=0–5



Calculating Lyapunov Exponent for (0,0,0), ρ=0–50



Lyapunov Exponent

Does the Lyapunov Exponent predict the time-dependent deviation between an orbit starting at the initial value $(0, 0, 0)$ and another orbit starting very close to $(0, 0, 0)$?

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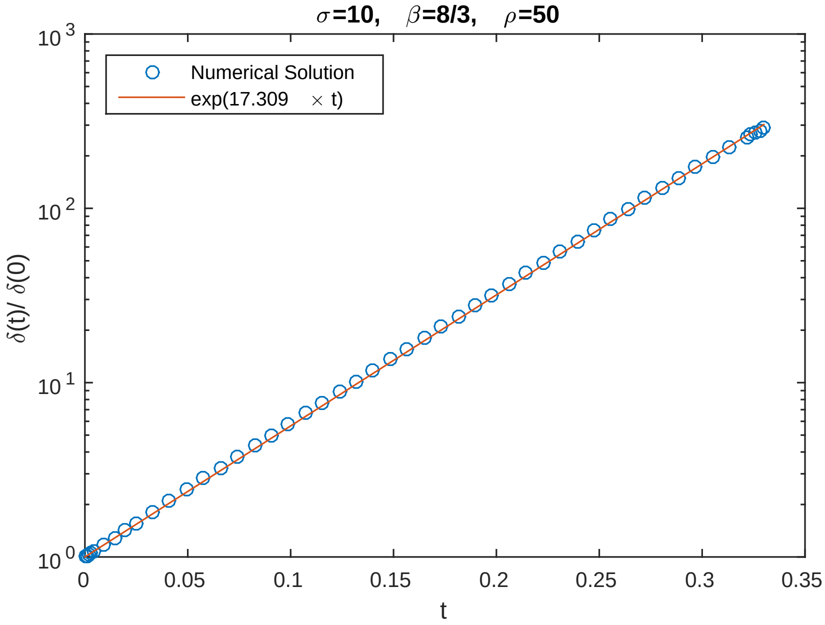
Let's look at $\rho = 50$

Reminder:

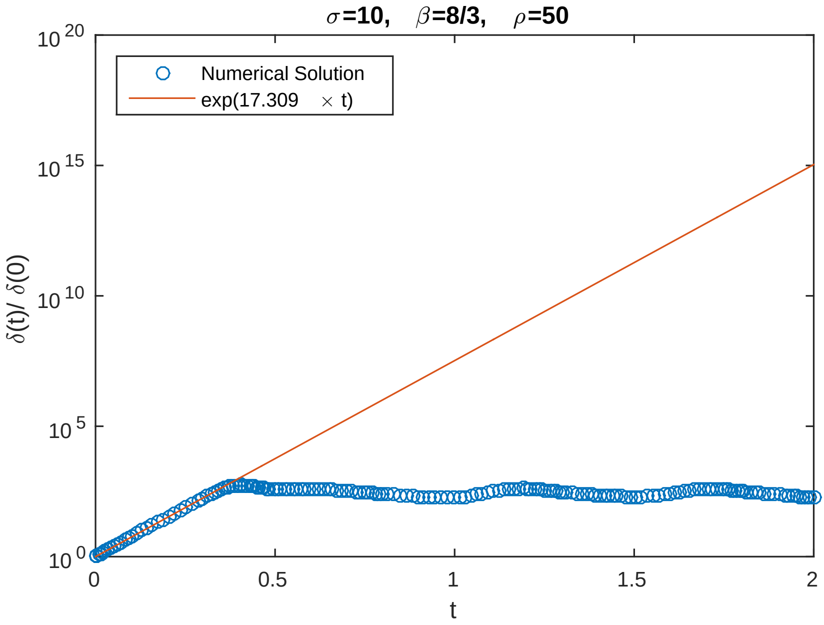
$$\delta(t) = b(t) - a(t)$$

$$|\delta(t)| \approx |\delta(0)|e^{\lambda t}$$

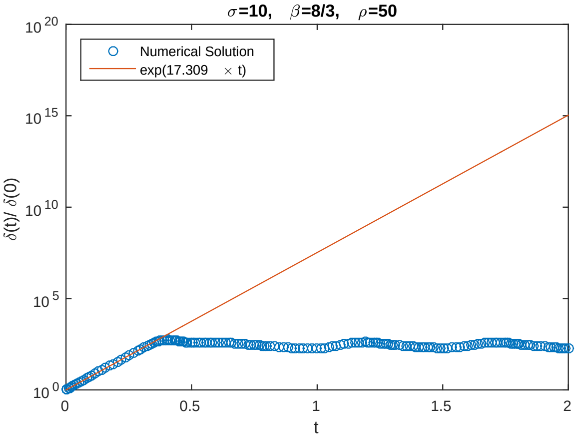
Lyapunov Exponent of Lorenz Equations



Lyapunov Exponent of Lorenz Equations

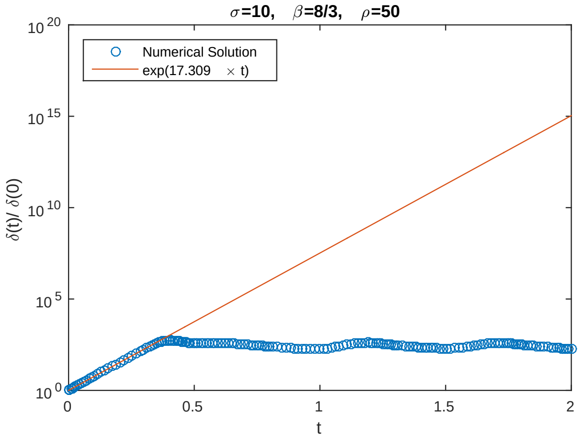


Lyapunov Exponent of Lorenz Equations



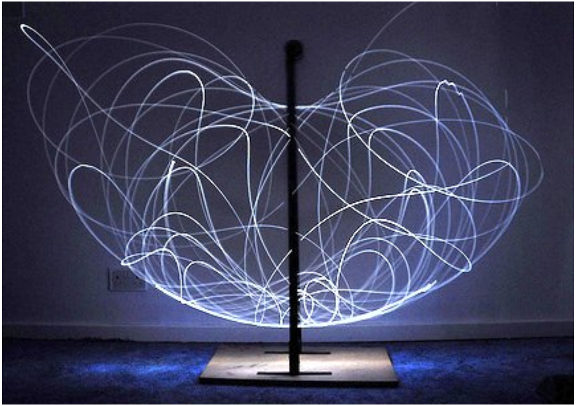
Solutions to the Lorenz equations occupy a finite volume of phase space…

Lyapunov Exponent of Lorenz Equations



Solutions to the Lorenz equations occupy a finite volume of phase space, so the two trajectories cannot deviate arbitrarily far apart. **The Lyapunov exponent does not tell the whole story.**

Lyapunov Exponent of Lorenz Equations



Analogously, a double pendulum can appear to behave chaotically but its motion is still constrained by its Hamiltonian. It can only occupy a finite region of phase space.

Lyapunov Exponent of Lorenz Equations

Lyapunov exponents (eigenvalues of Jacobian):

$$\lambda_1 = -\beta$$

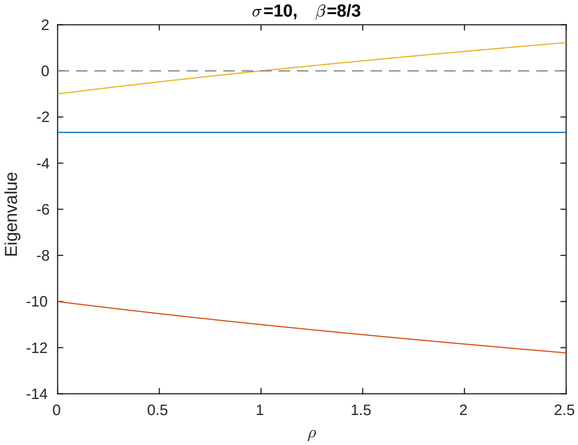
$$\lambda_2 = -\frac{\sigma}{2} + \frac{\sqrt{4\rho\sigma - 2\sigma + \sigma^2 + 1}}{2} - \frac{1}{2}$$

$$\lambda_3 = -\frac{\sigma}{2} - \frac{\sqrt{4\rho\sigma - 2\sigma + \sigma^2 + 1}}{2} - \frac{1}{2}$$

For $\rho < 1$ ($\sigma = 10$, $\beta = 8/3$) at critical point $(0, 0, 0)$, all three Lyapunov exponents are negative: $|\delta(t)| \approx |\delta(0)| e^{\lambda t} \rightarrow$ **no chaos**.

This means (0,0,0) is a **stable critical point** for $\rho < 1$.

Lyapunov Exponent of Lorenz Equations



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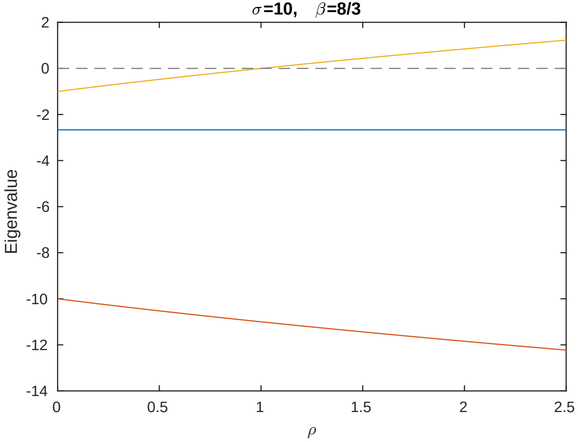

That is, the origin is a “sink” and all orbits with nearby starting points are drawn to it.

Lyapunov Exponent of Lorenz Equations

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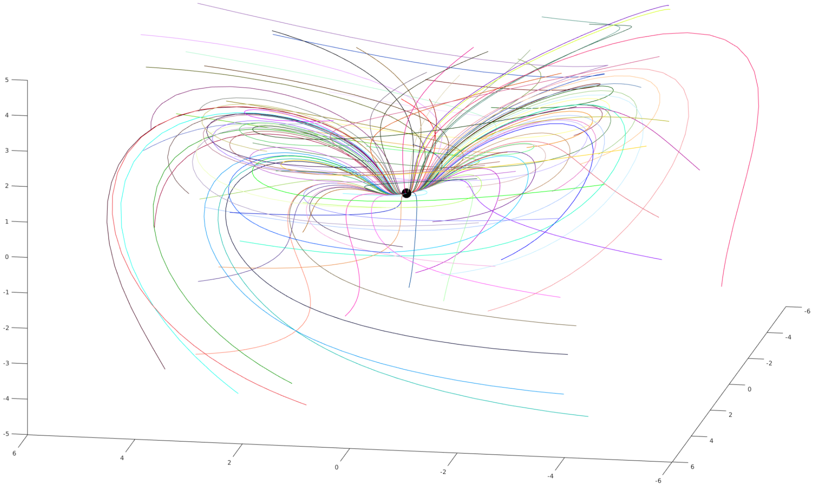

$\sigma = 10, \beta = 8/3, \rho = 0.5$

REMINDER: Critical Points

Lorenz Equations:

$$\begin{aligned} x' &= \sigma(y - x), \\ y' &= x(\rho - z) - y, \\ z' &= xy - \beta z. \end{aligned}$$

Critical points ($x' = y' = z' = 0$):

1. $(0, 0, 0)$
2. $(\sqrt{\beta(\rho - 1)}, \sqrt{\beta(\rho - 1)}, \rho - 1)$
3. $(-\sqrt{\beta(\rho - 1)}, -\sqrt{\beta(\rho - 1)}, \rho - 1)$

…if $\rho \geq 1$

…and only $(0, 0, 0)$ for $\rho \leq 1$.

Lyapunov Exponent of Lorenz Equations

- What about the other two critical points:

  $(\pm\sqrt{\beta(\rho-1)}, \pm\sqrt{\beta(\rho-1)}, \rho-1) = C^{\pm}$?

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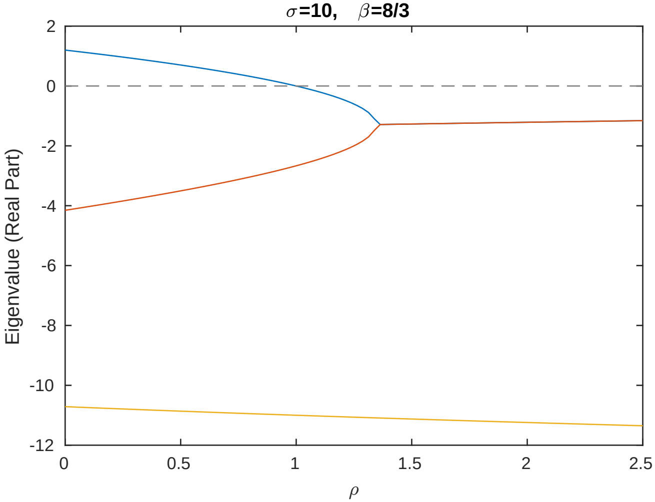
- When we do this, we find that for some values of ρ, the eigenvalues have an imaginary component.
- There are also bifurcations and other interesting behavior.

Lyapunov Exponent of Lorenz Equations: $C^{\pm}$



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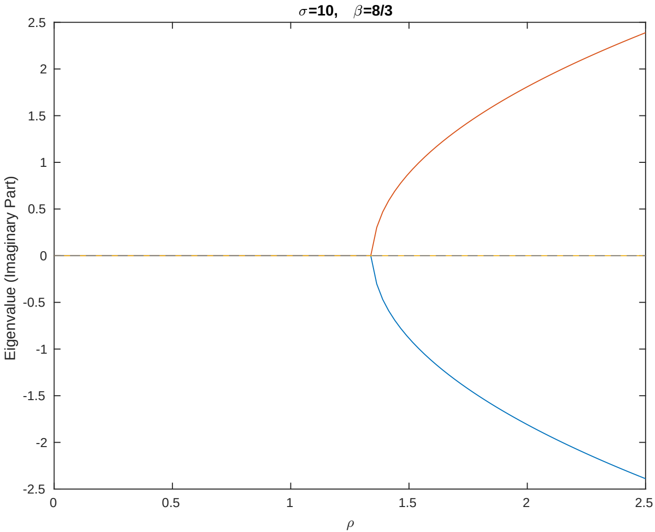

Lyapunov Exponent of Lorenz Equations

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So complex eigenvalues (λ) would mean…

oscillatory orbits (they begin at ρ=1.346) with a frequency given by the imaginary component.

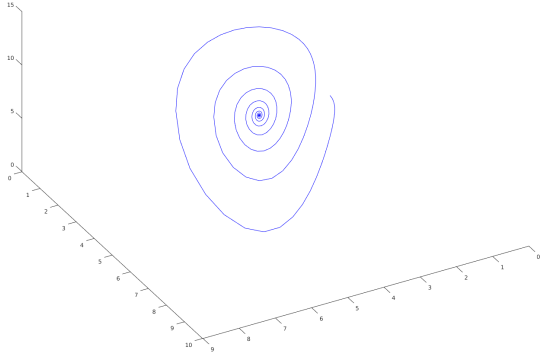
Types of Critical Points & Lyapunov Exponent

$$\lambda = A + Bi$$

A	**B**	**Type**
Negative	Zero	Stable
Negative	Non-zero	Attractor
Positive	Zero	Chaotic
Positive	Non-zero	Strange Attractor

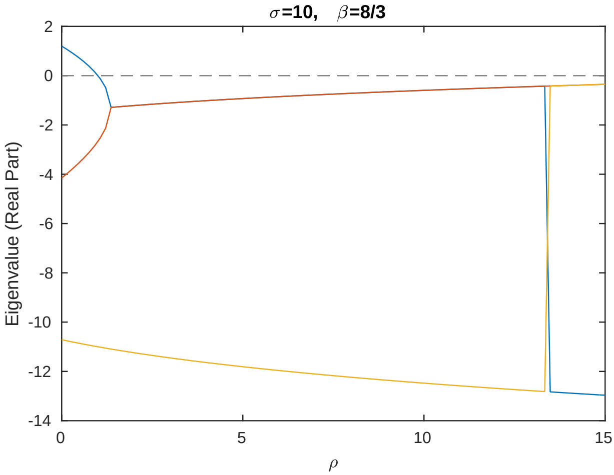
- **Stable:** if we start close, but not exactly on, the critical point, the solution will evolve toward the fixed point.
- **Attractor:** A region in space that is invariant under the evolution of time and attracts most, if not all, nearby trajectories.
- **Strange attractor:** An attractor that displays chaotic behavior (i.e., high sensitivity to initial conditions).

Lyapunov Exponent of Lorenz Equations



$\sigma = 10$, $\beta = 8/3$, $\rho = 10$. Orbits spiral towards either C^+ or C^-.

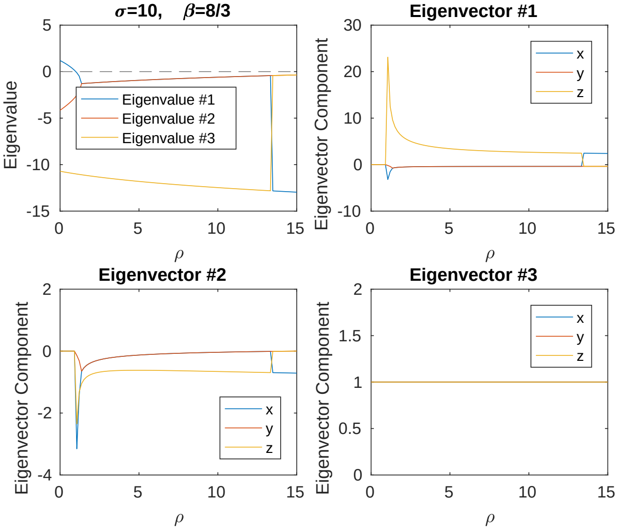
Lyapunov Exponent of Lorenz Equations



Plotting the eigenvalues for higher ρ, we see that at $\rho \approx 13.4$ two eigenvalues switch to each other's eigenvectors.

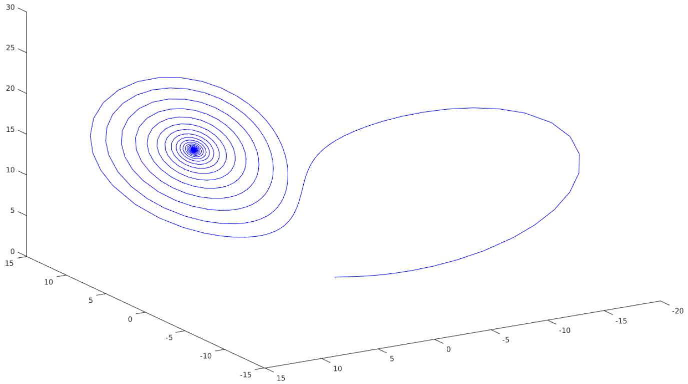
```
% Clear variables, close plots
clearvars;
close all

% Declare symbolic variables
syms x y z xp yp zp lambda
syms sigma_ positive
syms rho_ positive
syms beta_ positive

% Calculate the Jacobian
J=jacobian([sigma_*(y-x);x*(rho_-z)-y;x*y-beta_*z],[x,y,z]);

% Substitute the critical point C- into the Jacobian
J_1=subs(J,[x,y,z],[-sqrt(beta_*(rho_-1)),-sqrt(beta_*(rho_-1)),rho_-1]);

% [ -sigma_, sigma_,       0]
% [    rho_,     -1,       0]
% [       0,      0, -beta_]

% Calculate the eigenvalues of this Jacobian
[eigvecsm,eigvalsm]=eig(J_1);
eigvals=diag(eigvalsm);
%                                                                   -beta_
%  - sigma_/2 - (4*rho_*sigma_ - 2*sigma_ + sigma_^2 + 1)^(1/2)/2 - 1/2
%    (4*rho_*sigma_ - 2*sigma_ + sigma_^2 + 1)^(1/2)/2 - sigma_/2 - 1/2

% Plot the three eigenvalues for sigma=10, beta=8/3, and a range of rho
rho_vals=linspace(0,15,100);
eigvals_=zeros(numel(rho_vals),numel(eigvals));
eigvecs_=zeros(numel(rho_vals),numel(eigvals),3);
for ii=1:numel(rho_vals)
    eigvals_(ii,:)=double(subs(eigvals,[sigma_,beta_,rho_],[10,8./3.,rho_vals(ii)]));
    eigvecs_(ii,:,:)=double(subs(eigvecsm,[sigma_,beta_,rho_],[10,8./3.,rho_vals(ii)]));
end

subplot(2,2,1)
pp=plot(rho_vals,eigvals_,[0,max(rho_vals)],[0,0],'--');
set(pp(4),'Color',[0.5,0.5,0.5])
xlabel('\rho')
ylabel('Eigenvalue')
legend('Eigenvalue #1','Eigenvalue #2','Eigenvalue #3','Location','West')
title('\sigma=10, \beta=8/3')

subplot(2,2,2)
plot(rho_vals,squeeze(eigvecs_(:,1,[1,2,3])))
title('Eigenvector #1')
xlabel('\rho')
ylabel('Eigenvector Component')
legend('x','y','z');
```

Lyapunov Exponent of Lorenz Equations



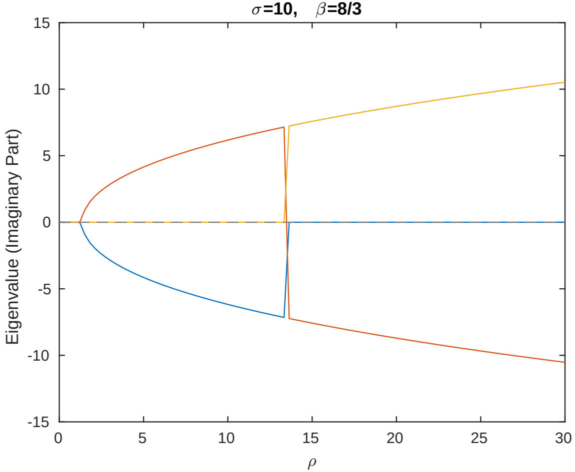
Each switched eigenvector also changes direction, and instead, a point starting closer to C^+ will eventually settle at C^- (and vice-versa).

Lyapunov Exponent of Lorenz Equations



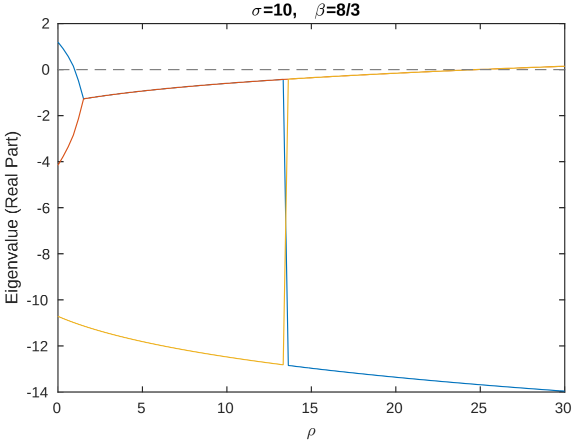
A point starting closer to C^+ will eventually settle at C^- (and vice-versa).

Lyapunov Exponent of Lorenz Equations



ρ=13–24: the imaginary part of the eigenvalues increase, meaning the frequency of orbits around $C^{\pm}$ increase. Solutions oscillate between C^{+} and C^{-} many times before finally spiraling into them.

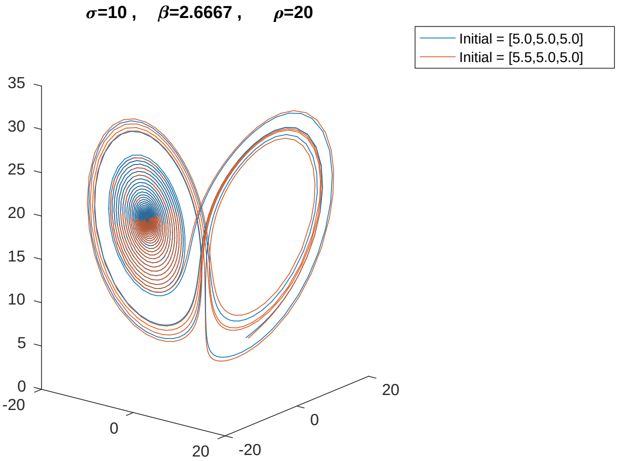
Lyapunov Exponent of Lorenz Equations



ρ=13–24: the real part of the eigienvalues for $C^{\pm}$ are still negative, so the solution will settle into either $C^{\pm}$ eventually, but the time it takes to do so can vary sensitively on the initial conditions.

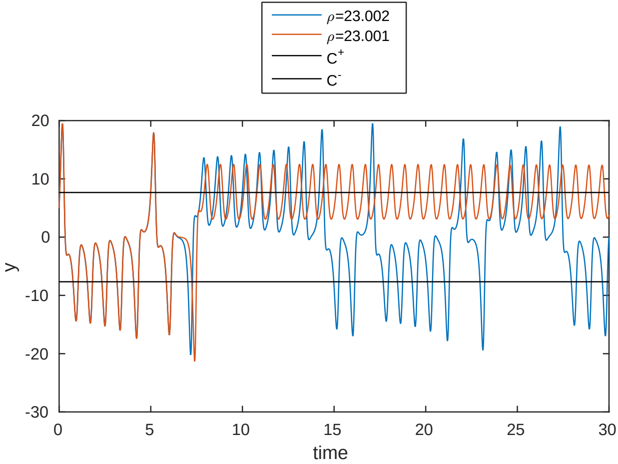
Problem #5

Problem #5: *Find a value of ρ (while keeping $\sigma = 10$ and $\beta = 8/3$) such that the solution does not depend sensitively on the initial values.*



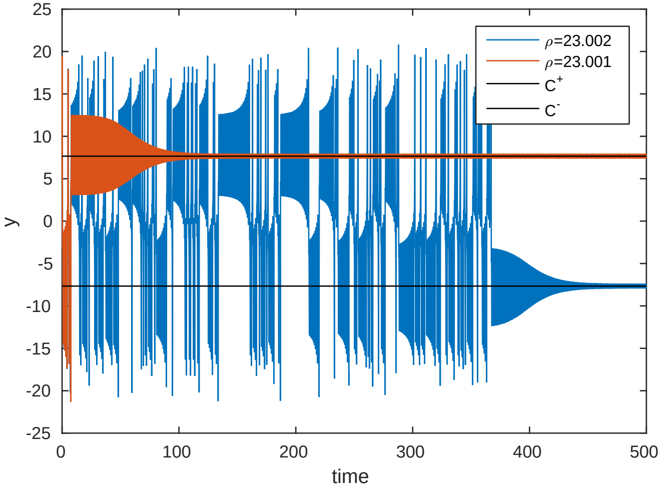
Solutions spirals into $C^{-} = (-7.1, -7.3, 19.0)$

Problem #5



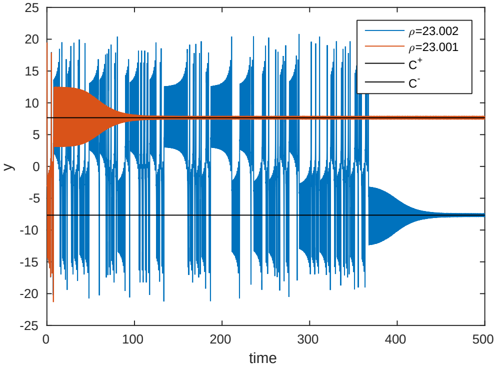
Some of you found interesting behavior in this region. Setting ρ=23.001 and ρ=23.002 and integrating $t = 0$–30 appeared to straddle a transition into chaos.

Problem #5



But if we integrate out to longer times, we see that each of these ρ values *eventually* spirals into $C^{\pm}$ (as they should because the real-part of the eigenvalues are still negative).

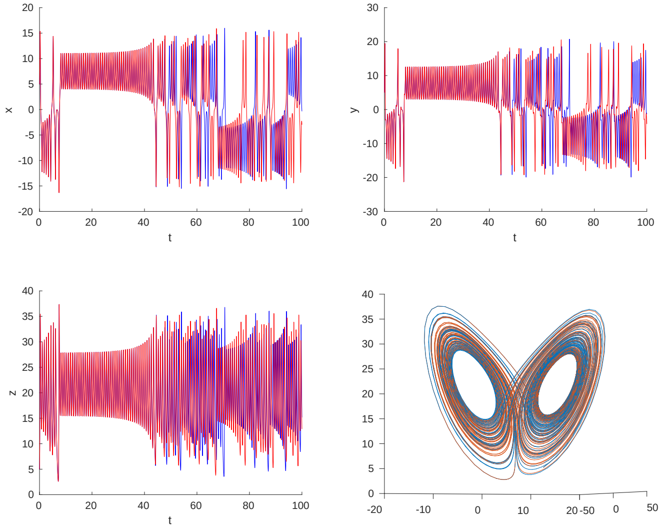
Problem #5



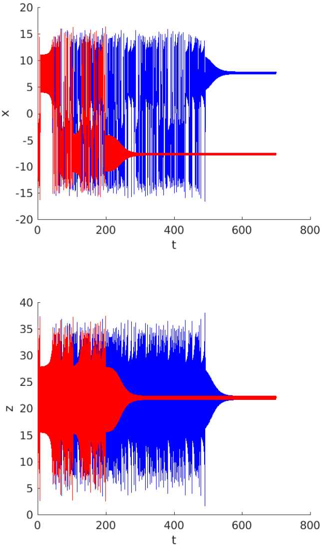
But if we integrate out to longer times, we see that each of these ρ values *eventually* spirals into $C^{\pm}$ (as they should because the real-part of the eigenvalues are still negative).

But the time a solution takes to do so (and whether it goes to C^+ or C^-) can vary *hugely* with small changes in ρ, initial conditions, or integration time step! *This is a fascinating region!*

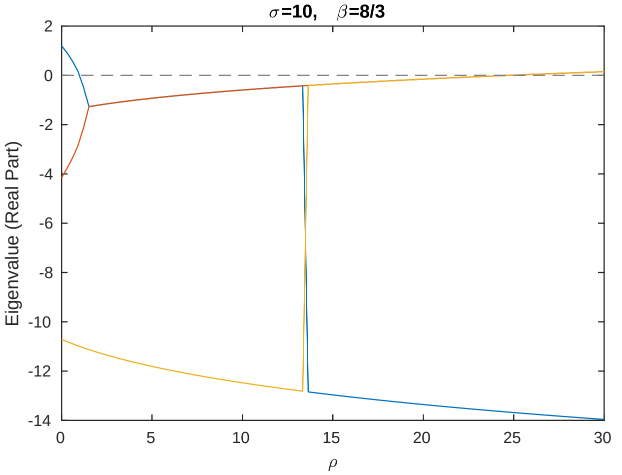
Problem #5: ρ=23.001100 & 23.001101



Problem #5: ρ=23.001100 & 23.001101

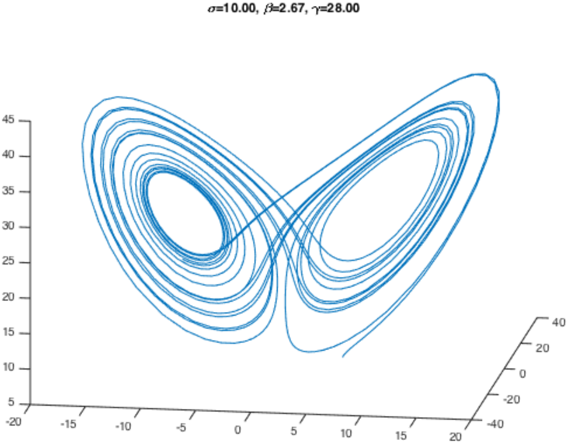


Lyapunov Exponent of Lorenz Equations



Around 24.74, one eigenvalue of $C^{\pm}$ turns positive again (Hopf bifurcation), so these attractors become chaotic, and because they have nonzero imaginary components, the orbits are spirals.

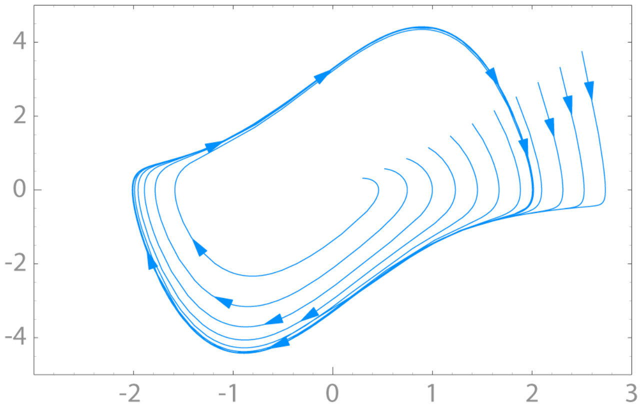
Lyapunov Exponent of Lorenz Equations



In this regime $C^{\pm}$ are strange attractors (displaying chaotic behavior). (This is the regime of the classic "Lorenz Attractor").

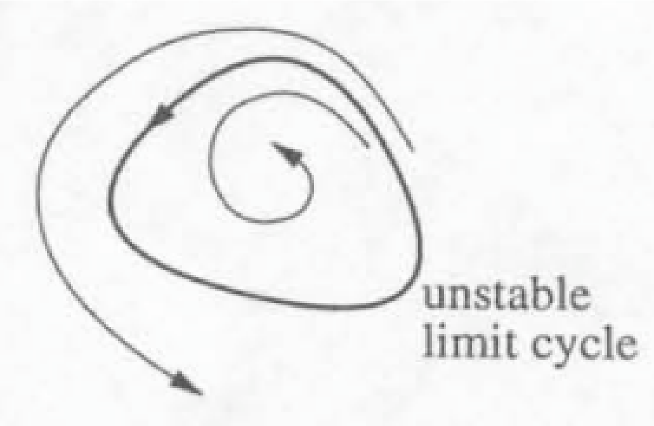
Limit Cycle

A limit cycle is an isolated closed trajectory. Limit cycles can be stable (nearby orbits spiral into it as $t \to \infty$)



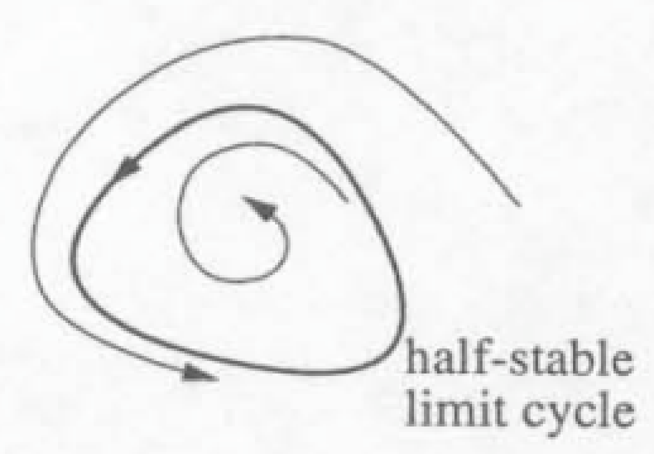
Limit Cycle

…unstable (nearby orbits spiral into it as $t \to -\infty$)



Limit Cycle

…or semi-stable, for example if the limit cycle is stable for trajectories approaching from inside, but unstable for trajectories approaching from outside.



Lyapunov Exponent of Lorenz Equations

There are many surprises in the parameter space, and many cannot be easily predicted:

- Point and chaotic attractors for $24.06 < \rho < 24.74$
- Strange limit cycle behavior at Hopf bifurcation
- The return of a global attracting limit cycle for intervals at large ρ
- $T(3, 2)$ torus knot at $\rho = 99.96$
- Lots more…

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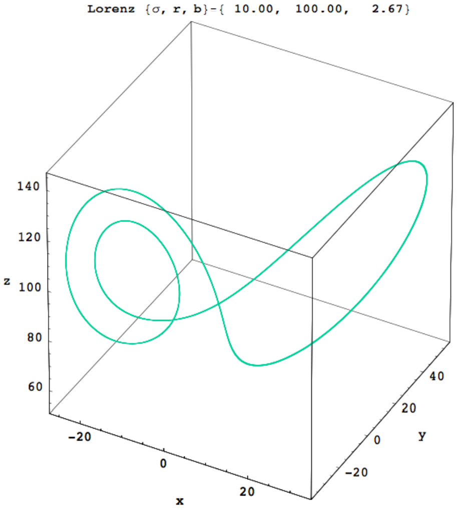
Exploring the phase space of the Lorenz equations is like a walk in the jungle.

Limit Cycle

Attractor is a stable limit cycle for ρ=100



Lecture Outline

- Lorenz Equations in an Analog Electronic Circuit
- Homework Review
- Chaotic Dynamics
- Lyapunov Exponent
- Lyapunov Exponent of the Lorenz Equations
- Chaotic motion in the Solar System

Chaotic motion in the Solar System

One of the reasons King Oscar wanted to know the solution to the n-body problem was to determine if the Solar System is stable.

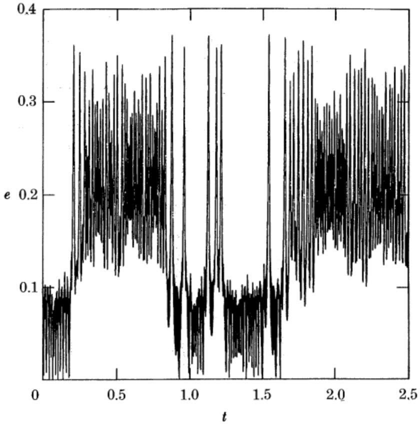


FIGURE 5. Eccentricity of a typical chaotic trajectory over a longer time interval. The time is now measured in millions of years. Bursts of high-eccentricity behaviour are interspersed with intervals of irregular low-eccentricity behaviour, broken by occasional spikes.

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- Arnold diffusion would cause more catastrophic chaos, like ejection of planets, but the timescales for this *appear* to be large.