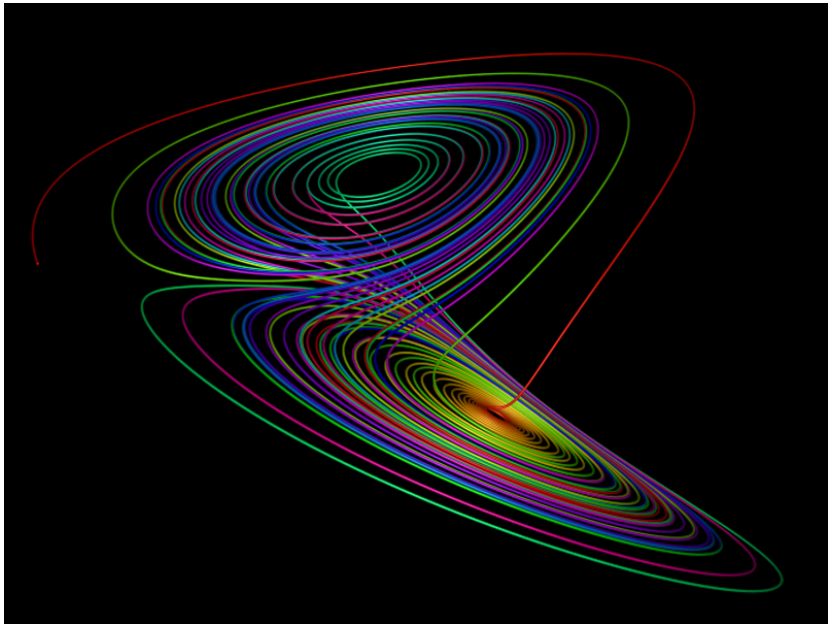# Physics 584 Computational Methods
The Lorenz Equations and Numerical Simulations of Chaos

Ryan Ogliore

March 1st, 2018

STRANGE,

YET ATTRACTIVE

Lecture Outline

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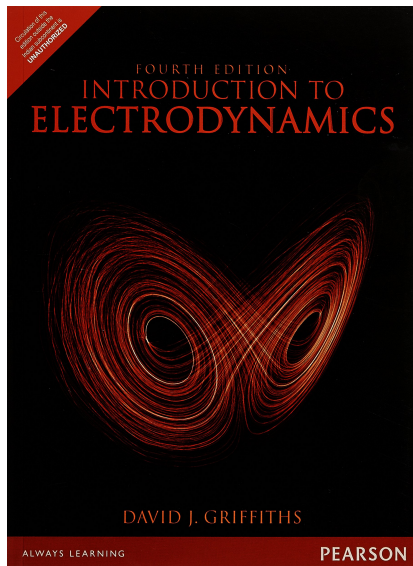
Lorenz Equations in an Analog Electronic Circuit

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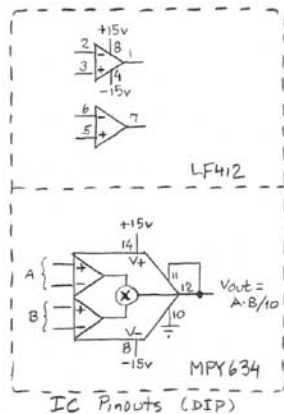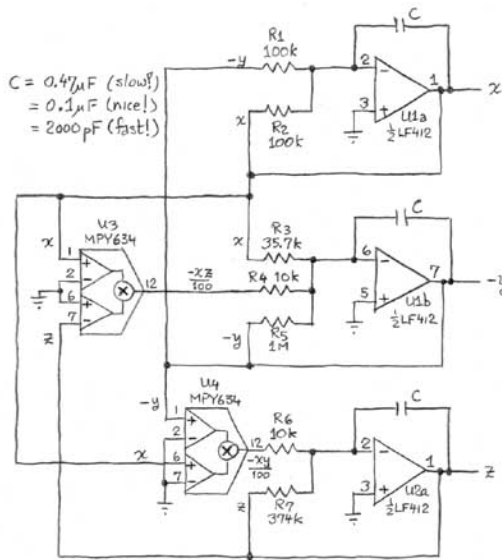
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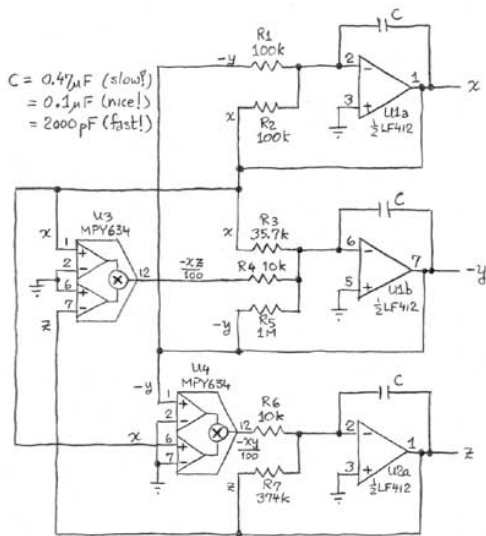
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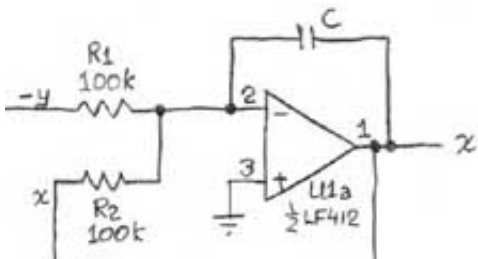
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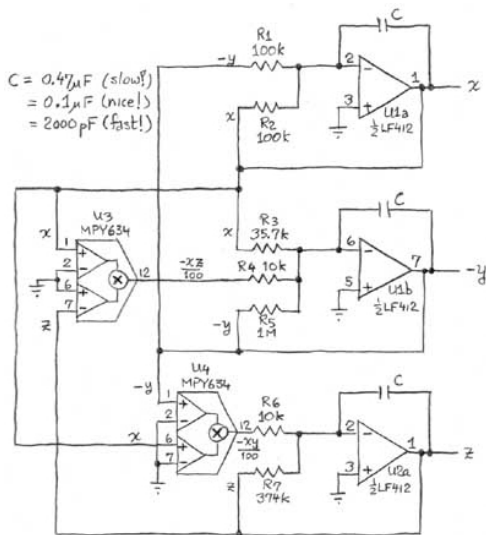
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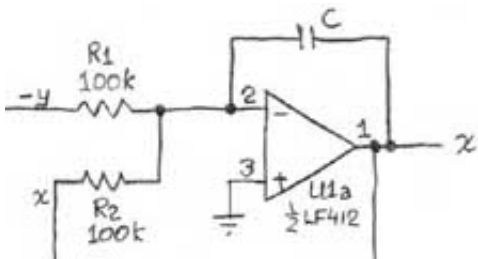
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- ► …and likewise for dy/dt and dz/dt in the Lorenz equations
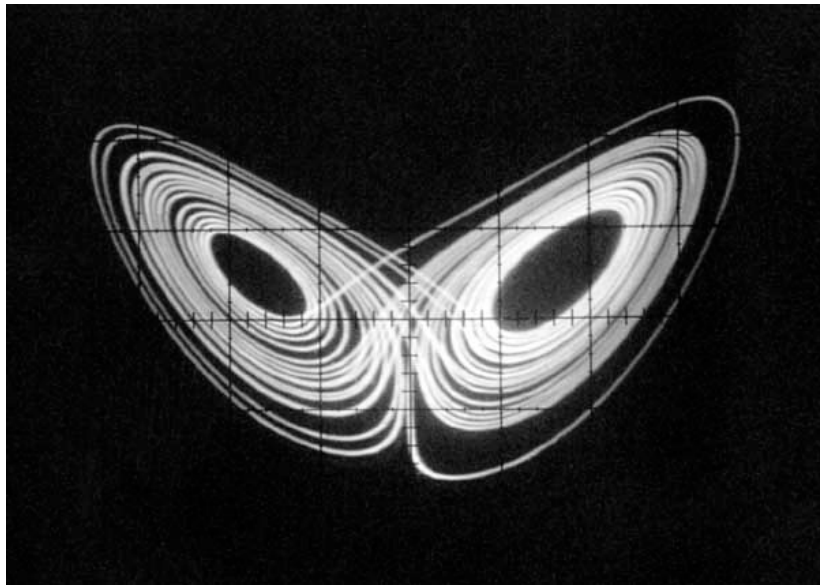
Lorenz Equations in an Analog Electronic Circuit

Lecture Outline

Homework (continued)

Edward Lorenz, a meterologist, created a simplified mathematical model for nonlinear atmospheric thermal convection in 1962. Lorenz's model frequently arises in other types of systems, e.g. dynamos and electrical circuits. Now known as the Lorenz equations, this model is a system of three ordinary differential equations:

$$
\begin{aligned}
\frac{dx}{dt} &= \sigma(y - x), \\
\frac{dy}{dt} &= x(\rho - z) - y, \\
\frac{dz}{dt} &= xy - \beta z.
\end{aligned}
$$

Note the last two equations involve quadratic nonlinearities. The intensity of the fluid motion is parameterized by the variable x; y and z are related to temperature variations in the horizontal and vertical directions.

Homework (continued)

Use Matlab's RK4 solver `ode45` to solve this system of ODEs with the following starting points and parameters.

1. With $\sigma = 1$, $\beta = 1$, and $\rho = 1$, solve the system of Lorenz Equations for $x(t = 0) = 1$, $y(t = 0) = 1$, and $z(t = 0) = 1$. Plot the orbit of the solution as a three-dimensional plot for times 0–100.

2. For the Earth's atmosphere reasonable values are $\sigma = 10$ and $\beta = 8/3$. Also set $\rho = 28$; and using starting values: $x(t = 0) = 5$, $y(t = 0) = 5$, and $z(t = 0) = 5$; solve the system of Lorenz Equations for $t = [0, 20]$. Plot the orbit of the solution as a three-dimensional plot for t=0–20. Also plot z vs. x. Do any of the orbits that appear to overlap in this plot actually overlap when viewed in the three-dimensional plot?

3. Plot x, y, and z vs. time on one graph using Matlab's `subplot` function.

Homework (continued)

4. Use the same parameters as in #2 but add a very small number (e.g. 10^{-6}) to one of the starting values. Plot x, y, and z vs. time for *both* of these curves (one red, one blue). Solve the equation for longer times to see when the two solutions diverge from each other.

5. Find a value of ρ (while keeping $\sigma = 10$ and $\beta = 8/3$) such that the solution does not depend sensitively on the initial values. Plot both curves for x, y, and z vs. time as you did in #4.

6. For ρ=70, $\sigma = 10$, $\beta = 8/3$, initial starting value (5,5,5), over a time range 0–50, calculate and plot one solution using the default maximum step size for ode45: $0.1 \times (t_{\text{final}} - t_{\text{initial}})$, and another solution for $1/1000^{\text{th}}$ of the default. Is this behavior related to the sensitivity on initial starting values you explored in #4?

Homework
`ode45`

- ► Matlab's `ode45` employs the Dormand-Prince method, a type of Runge-Kutta method
- ► This method computes 4th and 5th order solutions. **The difference between these solutions is then taken to be the error of the (fourth-order) solution**
- ► If the error is smaller than the tolerance, the step is successful
- ► If the error is larger than the *tolerance* (see `odeset`: `AbsTol` and `RelTol`) , the step is unsuccessful and the step size is decreased by an amount determined by the error/tolerance ratio
- ► This is an adaptive step-size integration algorithm. The user usually inputs a two-element vector for the time span over which to calculate the solution, the function returns a solution over it.
- ► `ode45` is an extremely accurate solver!

Homework

Regarding…

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The *Existence and Uniqueness Theorem* for systems of differential equations guarantees a unique solution for each set of initial conditions.

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The *Existence and Uniqueness Theorem* for systems of differential equations guarantees a unique solution for each set of initial conditions.

We will look into some of the other aspects of the Lorenz equations in today's lecture.

Lecture Outline

Lorenz Equations

Lorenz Equations:

$$\frac{dx}{dt} = \sigma(y - x),$$
$$\frac{dy}{dt} = x(\rho - z) - y,$$
$$\frac{dz}{dt} = xy - \beta z.$$

You explored the behavior of the solution for these equations with various parameters and starting conditions.

Some History

Isaac Newton

- ▶ The concept of the *Clockwork Universe* was accepted after Isaac Newton's laws of motion became canon (1690), and the improved analytical techniques for finding the equations for a system by Lagrange (1790) and Hamilton (1835).

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- Such a universe is completely determined by its initial conditions to evolve predictably with time (particularly championed by Laplace).

Pierre Laplace

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- ▶ Such a universe is completely determined by its initial conditions to evolve predictably with time (particularly championed by Laplace).

- ▶ Poincaré was the first person to see that Newton's laws of motion, in fact, predicted chaos all along.

Henri Poincaré

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- The problems were announced in *Acta Mathematica*, vol. 7, of 1885-1886. One problem was:

Given a system of arbitrarily many point masses that attract each other according to Newton's laws, under the assumption that no two points ever collide, find a representation of the coordinates of each point as a series solution in a variable that is some known function of time and for all of whose values the series converges uniformly.

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This is the n-body problem:

$$m_i \ddot{\mathbf{q}}_i = \sum_{j \neq i}^{n} \frac{G m_i m_j (\mathbf{q}_i - \mathbf{q}_j)}{|\mathbf{q}_i - \mathbf{q}_j|^3}, \ i = 1, \ldots, n$$

Some History

Poincaré proved that an analytical solution to the three-body problem was not possible.

"…it may happen that small differences in the initial conditions produce very great differences in the final phenomena. A small error in the former will produce an enormous error in the latter. Prediction becomes impossible…" – Henri Poincaré, 1892

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- However, it was overshadowed by quantum mechanics and relativity
- …until 1960, when computer simulations of simple systems of differential equations (e.g. the Lorenz equations) showed that even very simple systems can become chaotic
- Edward Lorenz: *"Chaos is when the present determines the future, but the approximate present does not approximately determine the future."*

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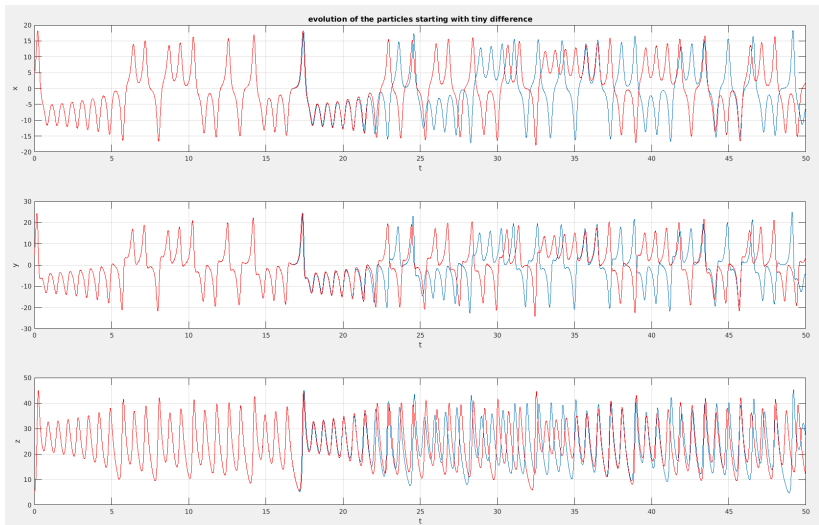- ▶ All memory that the two trajectories started out close is lost

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- ▶ For chaotic motion, two trajectories that are initially arbitrarily close in phase space will diverge exponentially in time from each other
- ▶ All memory that the two trajectories started out close is lost
- ▶ **Exponential divergence** is key. For nonchaotic motion, nearby trajectories diverge at most *linearly* with time

What is Deterministic Chaos? (Problem #4 in HW)

evolution of the particles starting with tiny difference

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For example, if the Solar System is chaotic, how long can we predict that the Earth is safe from being hit by a rogue planet/asteroid, or from being ejected from the Solar System?

Lecture Outline

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As the system evolves, we track $\delta(t)$.

The *maximal* Lyapunov exponent of the system is the number λ, if it exists, such that:

$$|\delta(t)| \approx |\delta(0)|e^{\lambda t}$$

Lyapunov Exponent

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The local Lyapunov exponents are the eigenvalues of the Jacobian of f (in $x' = f(x, t)$) at x_0.

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$$x' = \sigma(y - x),$$
$$y' = x(\rho - z) - y,$$
$$z' = xy - \beta z.$$

As you saw in the homework, an arbitrary starting point can diverge greatly from a very close neighboring starting point.

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Let's explore this by first looking at *critical points* (also called *fixed points*).

Critical Points

Lorenz Equations:

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Critical points are points in phase space that do not evolve. We will investigate the *stability* of fixed points because this will help us understand the general solutions of the system.

First, we find the **critical points** by setting $x' = y' = z' = 0$

(time derivatives are zero $\rightarrow$ no change in x, y, z with time).

Critical Points

Matlab code to find critical points (symbolic solution):

```matlab
1  clear all
2  close all
3
4  syms x y z xp yp zp
5  syms sigma_ positive
6  syms rho_ positive
7  syms beta_ positive
8
9  [x_c,y_c,z_c,~,conditions]=solve([0==sigma_*(y-x);0==x*(rho_-z)-y;0==x*y-beta_*z],...
10     'ReturnConditions', true, 'Real', true);
11
12  disp([x_c,y_c,z_c,conditions]);
13
14  % >>lyapunov_example
15  % [                              0,                             0,      0, rho < 1 | 1 <= rho]
16  % [    beta^(1/2)*(rho - 1)^(1/2),   beta^(1/2)*(rho - 1)^(1/2), rho - 1,          1 <= rho]
17  % [  -beta^(1/2)*(rho - 1)^(1/2),  -beta^(1/2)*(rho - 1)^(1/2), rho - 1,          1 <= rho]
```

Critical Points

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$$x' = \sigma(y - x),$$
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Critical points ($x' = y' = z' = 0$):

1. $(0, 0, 0)$
2. $(\sqrt{\beta(\rho - 1)}, \sqrt{\beta(\rho - 1)}, \rho - 1)$
3. $(-\sqrt{\beta(\rho - 1)}, -\sqrt{\beta(\rho - 1)}, \rho - 1)$

...if $\rho \geq 1$

...and only $(0, 0, 0)$ for $\rho < 1$.

Critical Points

Stable: if we start close, but not exactly on, the critical point, the solution will evolve toward the fixed point.

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Attractor: A region in space that is invariant under the evolution of time and attracts most, if not all, nearby trajectories.

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(The Lyapunov exponents are the eigenvalues of the Jacobian of the system.)

Calculating Lyapunov Exponent

The Jacobian matrix of the Lorenz system at the critical point $(0, 0, 0)$:

$$\mathbf{J} = \frac{d\mathbf{f}}{d\mathbf{x}} = \begin{bmatrix} \dfrac{\partial \mathbf{f}}{\partial x_1} & \cdots & \dfrac{\partial \mathbf{f}}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial f_m}{\partial x_1} & \cdots & \dfrac{\partial f_m}{\partial x_n} \end{bmatrix}$$

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$$\mathbf{J_1} = \begin{bmatrix} -\sigma & \sigma & 0 \\ \rho & -1 & 0 \\ 0 & 0 & -\beta \end{bmatrix}$$

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Which has the characteristic polynomial $(\det{(\lambda I - \mathbf{J_1})})$:

$$\lambda^3 + (\sigma + 1 + \beta)\lambda^2 + (\beta(\sigma + 1) + \sigma(1 - \rho))\lambda + \beta\sigma(1 - \rho)$$

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Which has the characteristic polynomial ($\det(\lambda I - \mathbf{J_1})$):

$$\lambda^3 + (\sigma + 1 + \beta)\lambda^2 + (\beta(\sigma + 1) + \sigma(1 - \rho))\lambda + \beta\sigma(1 - \rho)$$

Eigenvalues (Lyapunov exponents) are the roots of this polynomial:

Calculating Lyapunov Exponent

The Jacobian matrix of the Lorenz system at the critical point $(0, 0, 0)$:

$$\mathbf{J} = \frac{d\mathbf{f}}{d\mathbf{x}} = \begin{bmatrix} \dfrac{\partial \mathbf{f}}{\partial x_1} & \cdots & \dfrac{\partial \mathbf{f}}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial f_m}{\partial x_1} & \cdots & \dfrac{\partial f_m}{\partial x_n} \end{bmatrix}$$

$$\mathbf{J_1} = \begin{bmatrix} -\sigma & \sigma & 0 \\ \rho & -1 & 0 \\ 0 & 0 & -\beta \end{bmatrix}$$

Which has the characteristic polynomial ($\det(\lambda I - \mathbf{J_1})$):

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Eigenvalues (Lyapunov exponents) are the roots of this polynomial:

$$-\beta, \ -\sigma/2 \pm \left(\sqrt{4\rho\sigma - 2\sigma + \sigma^2 + 1}\right)/2 - 1/2$$

Calculating Lyapunov Exponent

Matlab code to calculate eigenvalues of the Jacobian (Lyapunov exponents) at the critical point $(0, 0, 0)$:

```matlab
% Clear variables , close plots
clear all
close all

% Declare symbolic variables
syms x y z xp yp zp lambda
syms sigma_ positive
syms rho_ positive
syms beta_ positive

% Calculate the Jacobian
J=jacobian([sigma_*(y-x);x*(rho_-z)-y;x*y-beta_*z],[x,y,z]);

% Substitute the first critical point (0,0,0) into the Jacobian
J_1=subs(J,[x,y,z],[0,0,0])

% [ -sigma_ , sigma_ ,       0]
% [   rho_ ,     -1,       0]
% [      0,      0, -beta_]

% Calculate the eigenvalues of this Jacobian
eigvals=eig(J_1);

%                                                            -beta_
% - sigma_/2 - (4*rho_*sigma_ - 2*sigma_ + sigma_^2 + 1)^(1/2)/2 - 1/2
%   (4*rho_*sigma_ - 2*sigma_ + sigma_^2 + 1)^(1/2)/2 - sigma_/2 - 1/2

% Plot the three eigenvalues for sigma=10, beta=8/3, and a range of rho
rho_vals=0:50;
eigvals_=zeros(numel(rho_vals),numel(eigvals));
for ii=1:numel(rho_vals)
    eigvals_(ii,:)=double(subs(eigvals,[sigma_,beta_,rho_],[10,8./3.,rho_vals(ii)]));
end

plot(rho_vals,eigvals_)
xlabel('\rho')
ylabel('Eigenvalue')
legend('Eigenvalue #1','Eigenvalue #2','Eigenvalue #3','Location','NorthWest')
title('\sigma=10, \beta=8/3')
pdfname='eigenvals_lorenz_jacobian.pdf';
print('-dpdf',pdfname);
[~,~]=system(['pdfcrop ' pdfname ' ' pdfname]);
```

Calculating Lyapunov Exponent for (0,0,0), ρ=0–5
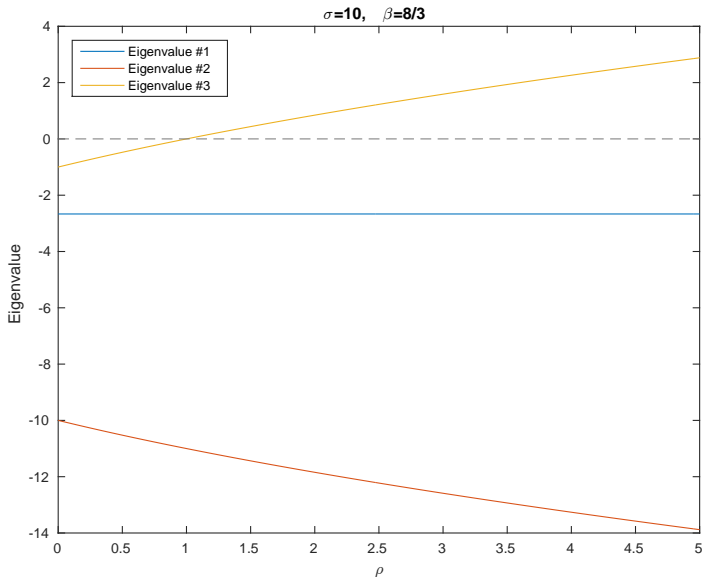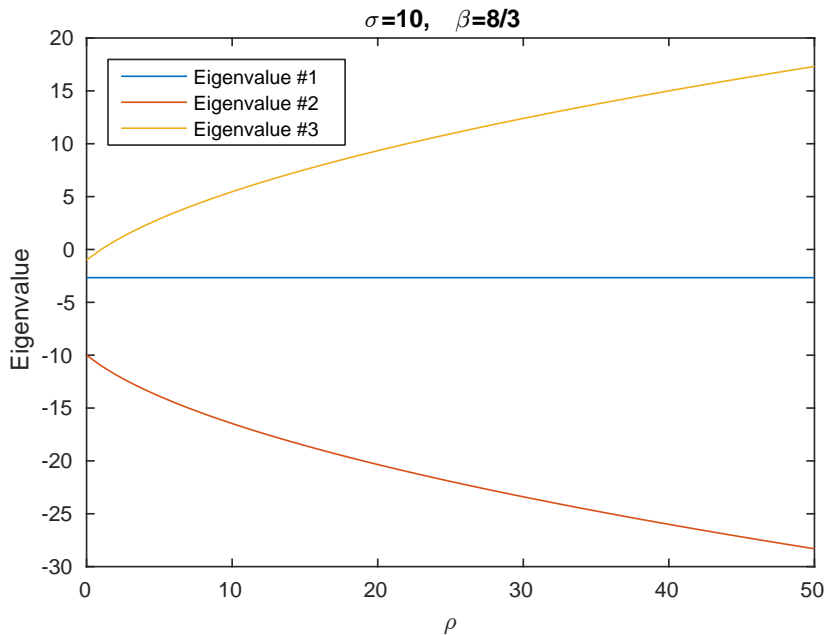
σ=10, β=8/3

Eigenvalue #1
Eigenvalue #2
Eigenvalue #3

Calculating Lyapunov Exponent for (0,0,0), ρ=0–50

σ**=10,** β**=8/3**

Lyapunov Exponent

Does the Lyapunov Exponent predict the time-dependent deviation between an orbit starting at the initial value $(0, 0, 0)$ and another orbit starting very close to $(0, 0, 0)$?

Lyapunov Exponent

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Let's look at $\rho = 50$

Lyapunov Exponent

Does the Lyapunov Exponent predict the time-dependent deviation
between an orbit starting at the initial value $(0, 0, 0)$ and another orbit
starting very close to $(0, 0, 0)$?

Let's look at $\rho = 50$

Reminder:
$$\delta(t) = b(t) - a(t)$$
$$|\delta(t)| \approx |\delta(0)|e^{\lambda t}$$

Lyapunov Exponent of Lorenz Equations

σ**=10,** β**=8/3,** ρ**=50**

Lyapunov Exponent of Lorenz Equations

σ=**10,** β=**8/3,** ρ=**50**

Lyapunov Exponent of Lorenz Equations

Solutions to the Lorenz equations occupy a finite volume of phase space, so the two trajectories cannot deviate arbitrarily far apart. **The Lyapunov exponent does not tell the whole story.**

Lyapunov Exponent of Lorenz Equations

Analogously, a double pendulum can appear to behave chaotically but its motion is still constrained by its Hamiltonian. It can only occupy a finite region of phase space.

Lyapunov Exponent of Lorenz Equations

Lyapunov exponents (eigenvalues of Jacobian):

$$\lambda_1 = -\beta$$

$$\lambda_2 = -\frac{\sigma}{2} + \frac{\sqrt{4\rho\sigma - 2\sigma + \sigma^2 + 1}}{2} - \frac{1}{2}$$

$$\lambda_3 = -\frac{\sigma}{2} - \frac{\sqrt{4\rho\sigma - 2\sigma + \sigma^2 + 1}}{2} - \frac{1}{2}$$

For $\rho < 1$ ($\sigma = 10$, $\beta = 8/3$) at critical point $(0,0,0)$, all three Lyapunov exponents are negative: $|\delta(t)| \approx |\delta(0)|e^{\lambda t} \rightarrow$ **no chaos**.

This means (0,0,0) is a **stable critical point** for $\rho < 1$.

Lyapunov Exponent of Lorenz Equations
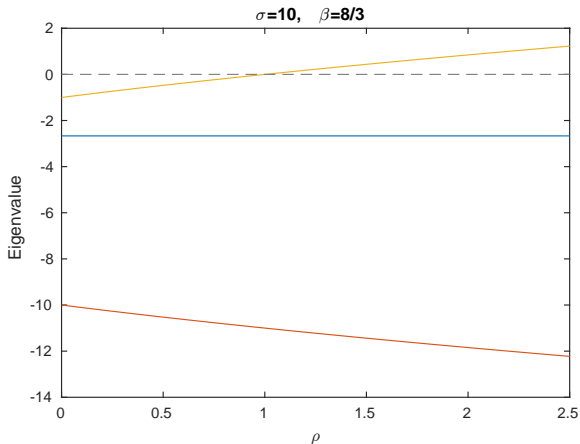
$\sigma=10, \quad \beta=8/3$

For $\rho < 1$ ($\sigma = 10$, $\beta = 8/3$) at critical point $(0, 0, 0)$, all three Lyapunov exponents are negative: $|\delta(t)| \approx |\delta(0)|e^{\lambda t} \to$ **no chaos**.

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Lyapunov Exponent of Lorenz Equations
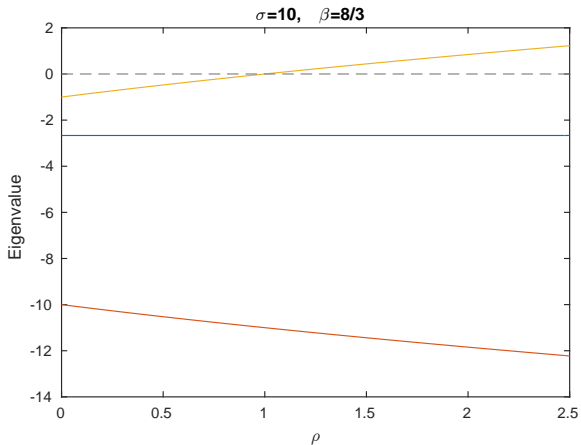
That is, the origin is a "sink" and all orbits with nearby starting points
are drawn to it.

Lyapunov Exponent of Lorenz Equations

The origin is a "sink" and all orbits with nearby starting points are drawn to it:
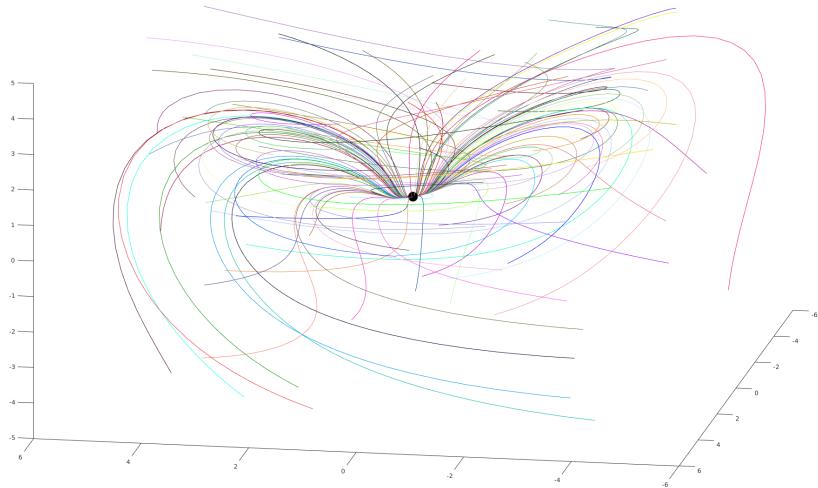
$\sigma = 10, \beta = 8/3, \rho = 0.5$

REMINDER: Critical Points

Lorenz Equations:

$$x' = \sigma(y - x),$$
$$y' = x(\rho - z) - y,$$
$$z' = xy - \beta z.$$

Critical points ($x' = y' = z' = 0$):

1. $(0, 0, 0)$
2. $(\sqrt{\beta(\rho - 1)}, \sqrt{\beta(\rho - 1)}, \rho - 1)$
3. $(-\sqrt{\beta(\rho - 1)}, -\sqrt{\beta(\rho - 1)}, \rho - 1)$

...if $\rho \geq 1$

...and only $(0, 0, 0)$ for $\rho \leq 1$.

Lyapunov Exponent of Lorenz Equations

What about the other two critical points:

$(\pm\sqrt{\beta(\rho-1)}, \pm\sqrt{\beta(\rho-1)}, \rho-1) = C^{\pm}$?

Are these stable (non-chaotic) or unstable (chaotic)? We'll need to calculate their Lyapunov exponents…

Lyapunov Exponent of Lorenz Equations

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When we do this, we find that for some values of ρ, the eigenvalues have an imaginary component!

Lyapunov Exponent of Lorenz Equations

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Are these stable (non-chaotic) or unstable (chaotic)? We'll need to calculate their Lyapunov exponents…

When we do this, we find that for some values of ρ, the eigenvalues have an imaginary component!

There are also bifurcations and other interesting behavior.

Lyapunov Exponent of Lorenz Equations: $C^{\pm}$
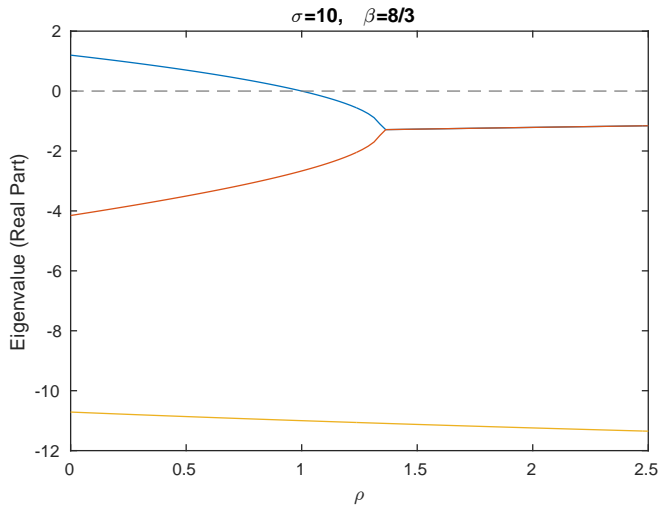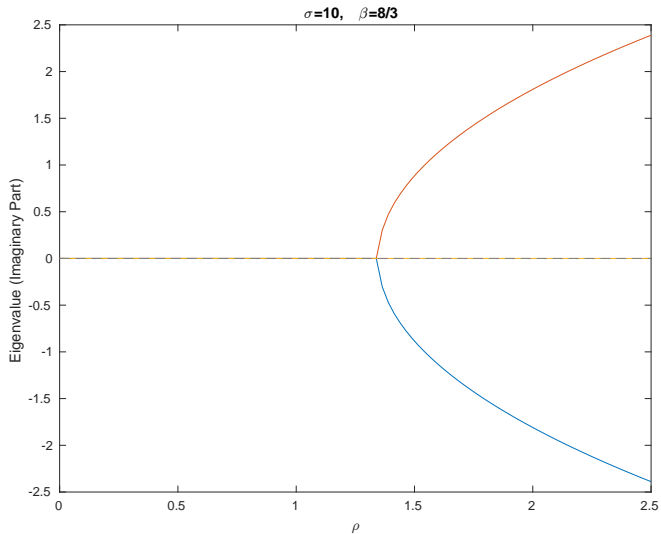
σ**=10,** β**=8/3**

Lyapunov Exponent of Lorenz Equations: $C^{\pm}$

σ=10, β=8/3

What does a complex eigenvalue mean for the orbits?

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Recall (the deviation between two orbits):

$$|\delta(t)| \approx |\delta(0)|e^{\lambda t}$$

Lyapunov Exponent of Lorenz Equations

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So complex eigenvalues (λ) would mean…

Lyapunov Exponent of Lorenz Equations

What does a complex eigenvalue mean for the orbits?

Recall (the deviation between two orbits):

$$|\delta(t)| \approx |\delta(0)|e^{\lambda t}$$

So complex eigenvalues (λ) would mean…*oscillatory* orbits (they begin at ρ=1.346) with a frequency given by the imaginary component.

Lyapunov Exponent of Lorenz Equations
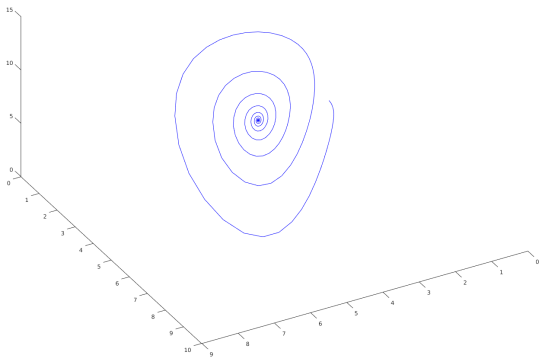
$\sigma = 10$, $\beta = 8/3$, $\rho = 10$. Orbits spiral towards either C^+ or C^-.

Lyapunov Exponent of Lorenz Equations
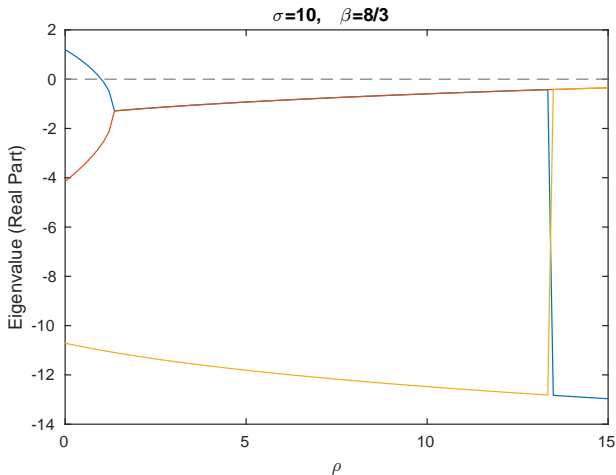
Plotting the eigenvalues for higher ρ, we see that at $\rho \approx 13.4$ two eigenvalues switch to each other's eigenvectors.

```matlab
% Clear variables, close plots
clear;
close all

% Declare symbolic variables
syms x y z xp yp zp lambda
syms sigma_ positive
syms rho_ positive
syms beta_ positive

% Calculate the Jacobian
J=jacobian([sigma_*(y-x);x*(rho_-z)-y;x*y-beta_*z],[x,y,z]);

% Substitute the critical point C- into the Jacobian
J_1=subs(J,[x,y,z],[-sqrt(beta_*(rho_-1)),-sqrt(beta_*(rho_-1)),rho_-1]);

% [ -sigma_,  sigma_,      0]
% [   rho_,      -1,       0]
% [     0,        0, -beta_]

% Calculate the eigenvalues of this Jacobian
[eigvecsm,eigvalsm]=eig(J_1);
eigvals=diag(eigvalsm);
%                                                          -beta_
%  - sigma_/2 - (4*rho_*sigma_ - 2*sigma_ + sigma_^2 + 1)^(1/2)/2 - 1/2
%    (4*rho_*sigma_ - 2*sigma_ + sigma_^2 + 1)^(1/2)/2 - sigma_/2 - 1/2

% Plot the three eigenvalues for sigma=10, beta=8/3, and a range of rho
rho_vals=linspace(0,15,100);
eigvals_=zeros(numel(rho_vals),numel(eigvals));
eigvecs_=zeros(numel(rho_vals),numel(eigvals),3);
for ii=1:numel(rho_vals)
    eigvals_(ii,:)=double(subs(eigvals,[sigma_,beta_,rho_],[10,8./3.,rho_vals(ii)]));
    eigvecs_(ii,:,:)=double(subs(eigvecsm,[sigma_,beta_,rho_],[10,8./3.,rho_vals(ii)]));
end

subplot(2,2,1)
pp=plot(rho_vals,eigvals_,[0,max(rho_vals)],[0,0],'--');
set(pp(4),'Color',[0.5,0.5,0.5])
xlabel('\rho')
ylabel('Eigenvalue')
legend('Eigenvalue #1','Eigenvalue #2','Eigenvalue #3','Location','West')
title('\sigma=10, \beta=8/3')

subplot(2,2,2)
plot(rho_vals,squeeze(eigvecs_(:,1,[1,2,3])))
title('Eigenvector #1')
xlabel('\rho')
ylabel('Eigenvector Component')
```

Lyapunov Exponent of Lorenz Equations
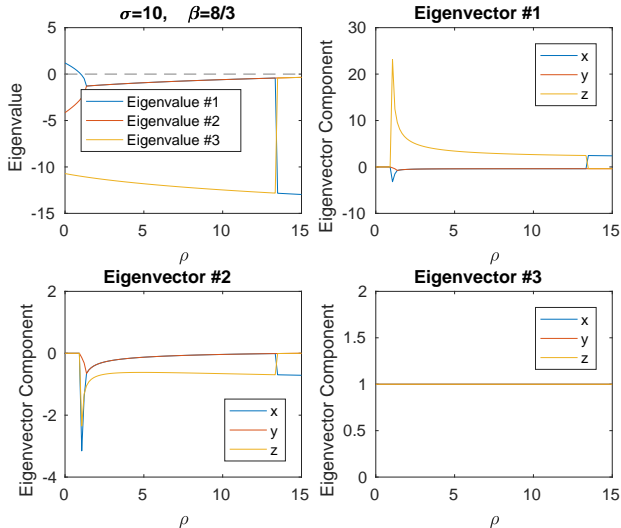
Each switched eigenvector also changes direction, and instead, a point starting closer to C^+ will eventually settle at C^- (and vice-versa).

Lyapunov Exponent of Lorenz Equations
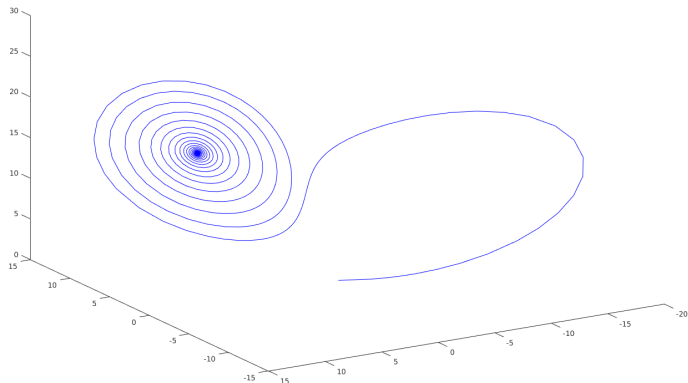
A point starting closer to C^+ will eventually settle at C^- (and vice-versa).

Lyapunov Exponent of Lorenz Equations
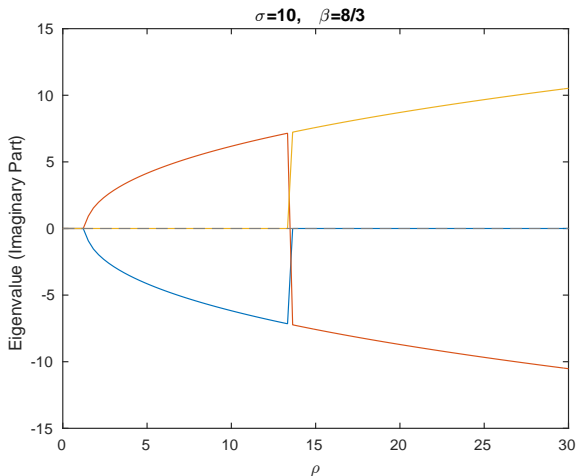
ρ=13–24: the imaginary part of the eigenvalues increase, meaning the frequency of orbits around $C^{\pm}$ increase. Solutions oscillate between C^+ and C^- many times before finally spiraling into them.

Lyapunov Exponent of Lorenz Equations

σ=10, β=8/3
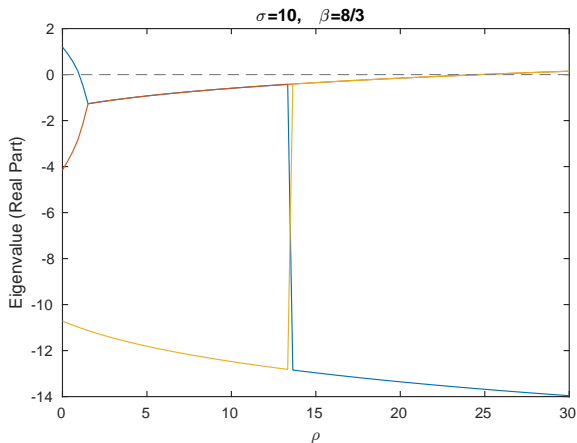
ρ=13–24: the real part of the eigienvalues for $C^{\pm}$ are still negative, so the solution will settle into either $C^{\pm}$ eventually, but the time it takes to do so can vary sensitively on the initial conditions.

Problem #5

Problem #5: *Find a value of ρ (while keeping $\sigma = 10$ and $\beta = 8/3$) such that the solution does not depend sensitively on the initial values.*

$\sigma=10$, $\beta=2.6667$, $\rho=20$

Legend:
- Initial = [5.0,5.0,5.0]
- Initial = [5.5,5.0,5.0]
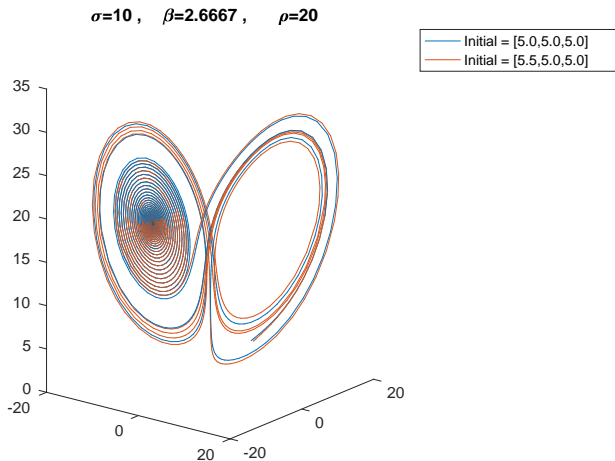
Solutions spirals into $C^- = (-7.1, -7.3, 19.0)$

Problem #5
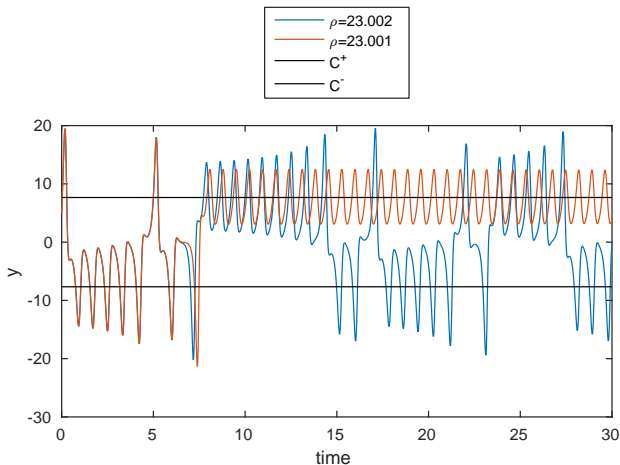
Some of you found interesting behavior in this region. Setting ρ=23.001 and ρ=23.002 and integrating $t = 0$–30 appeared to straddle a transition into chaos.

Problem #5
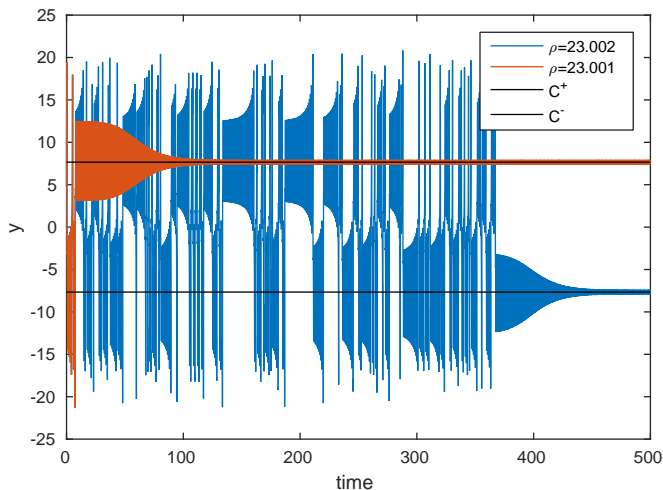
But if we integrate out to longer times, we see that each of these ρ values eventually spirals into $C^{\pm}$ (as they should because the real-part of the eigenvalues are still negative).

Problem #5
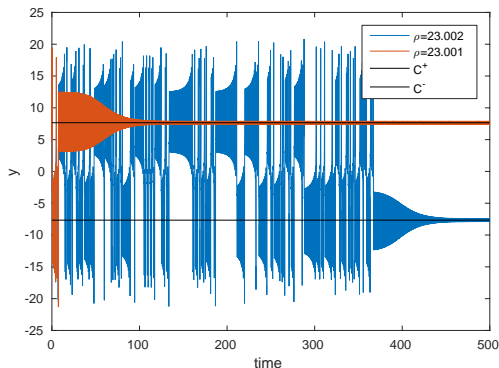
But if we integrate out to longer times, we see that each of these ρ values eventually spirals into $C^{\pm}$ (as they should because the real-part of the eigenvalues are still negative).

But the time a solution takes to do so (and whether it goes to C^+ or C^-) can vary *hugely* with small changes in ρ, initial conditions, or integration time step! *This is a fascinating region!*
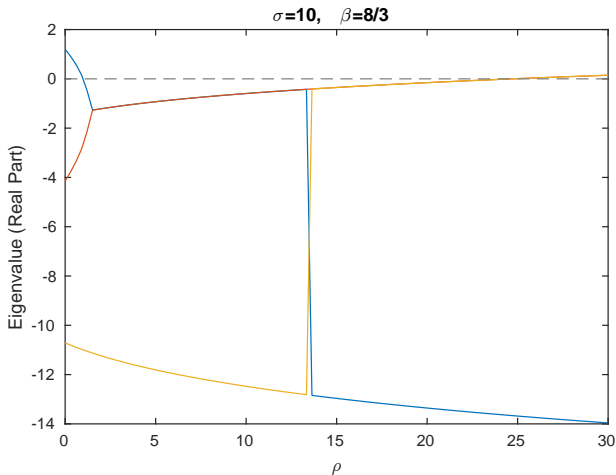
Problem #5: ρ=23.001100 & 23.001101
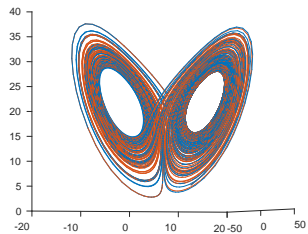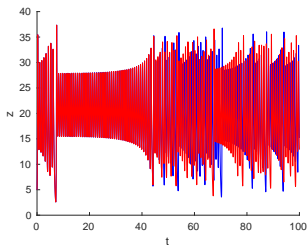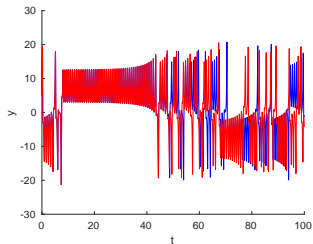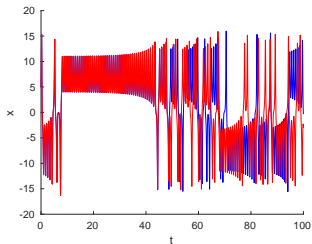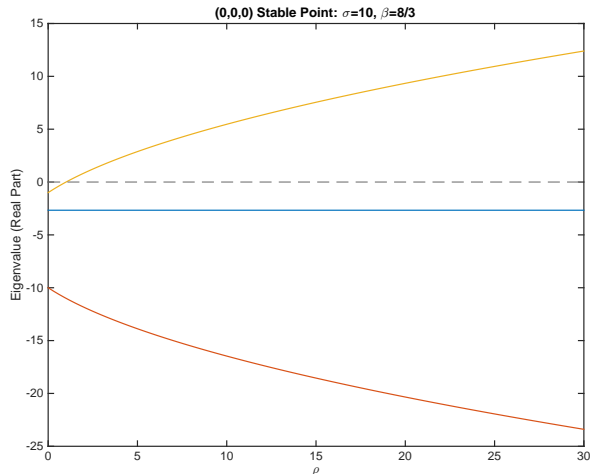
Problem #5: ρ=23.001100 & 23.001101

Lyapunov Exponent of Lorenz Equations

σ=10, β=8/3
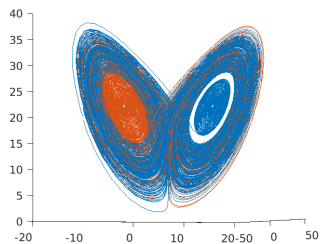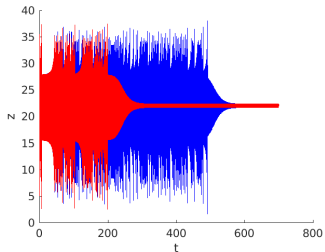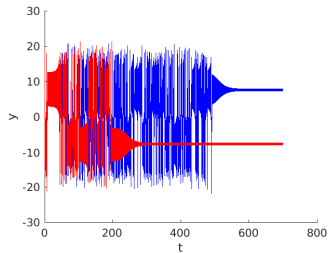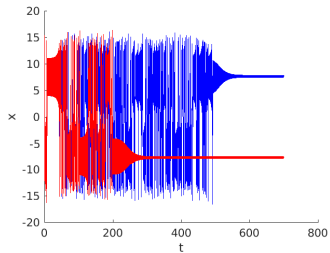
Around 24.74, one eigenvalue of $C^{\pm}$ turns positive again (Hopf bifurcation), so these attractors become chaotic, and because they have nonzero imaginary components, the orbits are spirals.

Lyapunov Exponent of Lorenz Equations

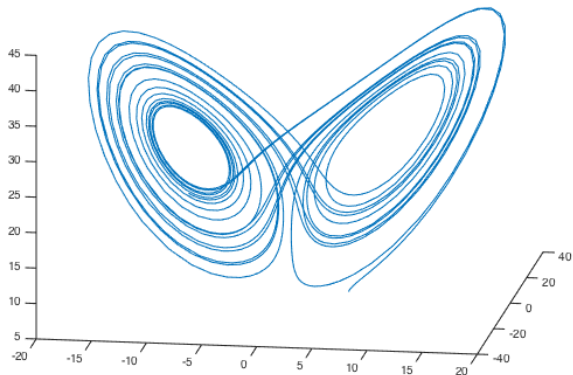**(0,0,0) Stable Point: σ=10, β=8/3**

The origin (0,0,0) remains the same: unstable with interesting manifolds that depend on ρ (no spiral because it does not have an imaginary component in its eigenvalues).
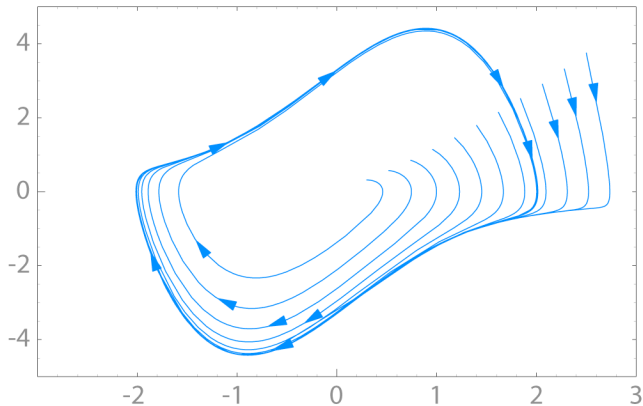
Lyapunov Exponent of Lorenz Equations

$\sigma=10.00,\ \beta=2.67,\ \gamma=28.00$

In this regime $C^{\pm}$ are strange attractors (displaying chaotic behavior).
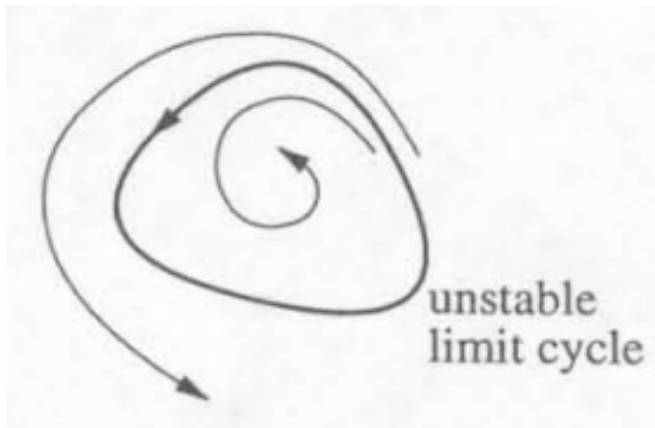(This is the regime of the classic "Lorenz Attractor").

Limit Cycle

A limit cycle is an isolated closed trajectory. Limit cycles can be stable (nearby orbits spiral into it as $t \to \infty$, see below Van der Pol oscillator)
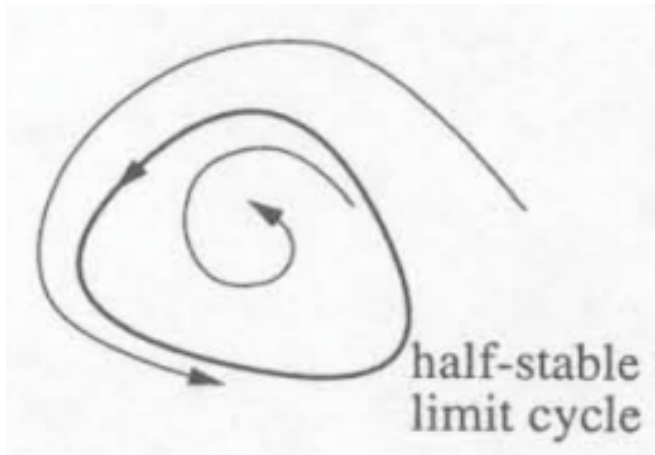
Limit Cycle

…unstable (nearby orbits spiral into it as $t \to -\infty$)

unstable
limit cycle

Limit Cycle

…or semi-stable, for example if the limit cycle is stable for trajectories approaching from inside, but unstable for trajectories approaching from outside.

half-stable limit cycle

Lyapunov Exponent of Lorenz Equations

There are many surprises in the parameter space, and many cannot be easily predicted:

- Point and chaotic attractors for $24.06 < \rho < 24.74$
- Strange limit cycle behavior at Hopf bifurcation
- The return of a global attracting limit cycle for intervals at large ρ
- Lots more…

Lyapunov Exponent of Lorenz Equations

There are many surprises in the parameter space, and many cannot be easily predicted:

- Point and chaotic attractors for $24.06 < \rho < 24.74$
- Strange limit cycle behavior at Hopf bifurcation
- The return of a global attracting limit cycle for intervals at large ρ
- Lots more…

Exploring the phase space of the Lorenz equations is like a walk in the jungle.

Limit Cycle

Attractor is a stable limit cycle for $\rho=100$
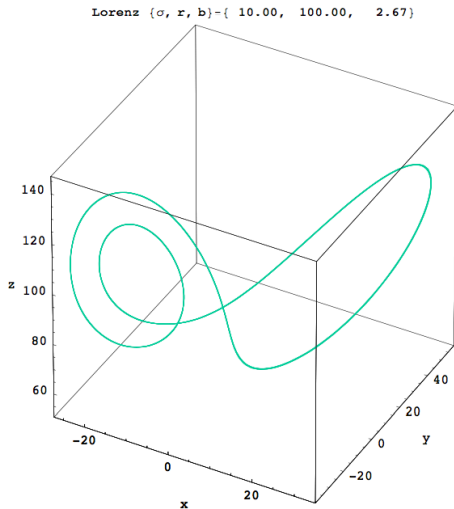
Lorenz $\{\sigma, r, b\}-\{\ 10.00,\ \ 100.00,\ \ \ 2.67\}$

Lecture Outline

Chaotic motion in the Solar System

One of the reasons King Oscar wanted to know the solution to the n-body problem was to determine if the Solar System is stable.
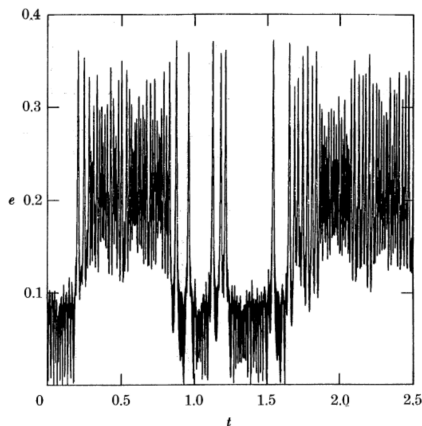

FIGURE 5. Eccentricity of a typical chaotic trajectory over a longer time interval. The time is now measured in millions of years. Bursts of high-eccentricity behaviour are interspersed with intervals of irregular low-eccentricity behaviour, broken by occasional spikes.

Chaotic motion in the Solar System

- ▶ Asteroids in orbital resonances with Jupiter can be sent into Earth-crossing orbits.

Chaotic motion in the Solar System

- Asteroids in orbital resonances with Jupiter can be sent into Earth-crossing orbits.

- Pluto has a Lyapunov exponent of $1/20\ \text{Myr}^{-1}$. This implies a *Lyapunov time scale* of ~ 20 Myr before neighboring orbits diverge significantly (i.e. it's not possible to determine Pluto's orbit precisely ~ 100 Myr from now).

Chaotic motion in the Solar System

- ▶ Asteroids in orbital resonances with Jupiter can be sent into Earth-crossing orbits.
- ▶ Pluto has a Lyapunov exponent of $1/20$ Myr^{-1}. This implies a *Lyapunov time scale* of $\sim$20 Myr before neighboring orbits diverge significantly (i.e. it's not possible to determine Pluto's orbit precisely $\sim$100 Myr from now).
- ▶ A rocket launch will change the Earth's position by $\sim$0.5° after 100 Myr.

Chaotic motion in the Solar System

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- ▶ A rocket launch will change the Earth's position by $\sim 0.5°$ after 100 Myr.
- ▶ But the Solar Sytem's dynamics are governed by a Hamiltonian, so chaos can only be local (a specific area of phase space).

Chaotic motion in the Solar System

- ▶ Asteroids in orbital resonances with Jupiter can be sent into Earth-crossing orbits.
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- ▶ A rocket launch will change the Earth's position by $\sim$0.5° after 100 Myr.
- ▶ But the Solar Sytem's dynamics are governed by a Hamiltonian, so chaos can only be local (a specific area of phase space).
- ▶ Arnold diffusion (nonconservation of invariant quantities) would allow for local chaos to diffuse throughout the phase space.

Chaotic motion in the Solar System

- ▶ Asteroids in orbital resonances with Jupiter can be sent into Earth-crossing orbits.
- ▶ Pluto has a Lyapunov exponent of $1/20$ Myr^{-1}. This implies a *Lyapunov time scale* of $\sim$20 Myr before neighboring orbits diverge significantly (i.e. it's not possible to determine Pluto's orbit precisely $\sim$100 Myr from now).
- ▶ A rocket launch will change the Earth's position by $\sim 0.5°$ after 100 Myr.
- ▶ But the Solar Sytem's dynamics are governed by a Hamiltonian, so chaos can only be local (a specific area of phase space).
- ▶ Arnold diffusion (nonconservation of invariant quantities) would allow for local chaos to diffuse throughout the phase space.
- ▶ Arnold diffusion would cause more catastrophic chaos, like ejection of planets, but the timescales for this *appear* to be large.